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Remarcable Measurable Functions

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Abstract

We present some measurable functions, specially Lebesgue measurable functions.

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1 Remarkable classes of sets and positive measures

In the following we define several classes of sets: algebra of sets, σ-algebra of sets, σ-algebra of Borel sets on a topological space.

Definition 1. A class of sets on X is a part nonempty of $\mathcal{P}(X)$.

Definition 2. A class of sets $\mathcal{R}$ on X it is called ring of sets, provided that, from $A, B \in \mathcal{R}$ it follows that $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$.

Definition 3. A class of sets $\mathcal{A}$ will be called algebra of sets provided that it holds:

1. $X \in \mathcal{A}$
2. For any $A, B \in \mathcal{A}$ we have $A \cup B \in \mathcal{A}$ and $A \setminus B \in \mathcal{A}$.

Definition 4. An $\mathcal{A}$ algebra will be called σ algebra if for any family $(A_n)_{n\geq 1}$ included in $\mathcal{A}$ it follows that $\cup_{n\geq 1} A_n \in \mathcal{A}$.

It's clear that the set of parts of X (i.e. $\mathcal{P}(X)$) is a σ-algebra of sets and intersection of a family of σ-algebras is also a σ-algebra of sets.

If $\mathcal{A}$ is a class of sets we denote by $\sigma(\mathcal{A})$ the intersection of the family of σ-algebras of sets including $\mathcal{A}$. This class of sets is the smallest σ-algebra of sets including $\mathcal{A}$ and is called σ-algebra generated by $\mathcal{A}$.

Definition 5. Let X be a topological space and $\mathcal{T}(\mathcal{F})$ the open (resp. closed) sets of X. We denote by $\mathcal{B}(X)$ the σ-algebra generated by $\mathcal{T}$. Elements of $\mathcal{B}(X)$ is called the Borel sets on X. Obviously $\mathcal{F} \subset \mathcal{B}(X)$ and $\mathcal{B}(X)$ is also the σ-algebra generated by $\mathcal{F}$.

2 Positive measures

Let X be a set and let $\mathcal{A}$ be a σ-algebra on X.

Definition 6. A function $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is called a positive measure if $\mu(\phi) = 0$ and for any sequence $(A_n)_n$ included in $\mathcal{A}$ such that $A_n \cap A_m = \emptyset$ whenever $n \neq m$ we have that $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

We denote by $\mathcal{M}_+(\mathcal{A})$ the set of positive measures on $\mathcal{A}$. From the definition the following assertions hold:

1. μ is finite additive, i.e. if $A, B \in \mathcal{A}$, such that $A \cap B = \phi$ it follows that $\mu(A \cup B) = \mu(A) + \mu(B)$.
2. μ is increasing. i.e. $A, B \in \mathcal{A}$, such that $A \subset B$ it follows that $\mu(A) \leq \mu(B)$, and if $\mu(A) < \infty$ then we get $\mu(B \setminus A) = \mu(B) - \mu(A)$.

Proposition 1. Let $\mu \in \mathcal{M}_+(\mathcal{A})$ be and $(A_n)_n$ a sequence of $\mathcal{A}$. The following assertions hold:

a) $A_n \uparrow A$, $A \in \mathcal{A}$ it follows that $\mu(A_n) \uparrow \mu(A)$;

b) $A_n \downarrow A$, $A \in \mathcal{A}$ and $\inf_n \mu(A_n) < +\infty$ it follows that $\mu(A_n) \downarrow \mu(A)$.

Remark 1. If $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is a finite additive function then affirmation a) from above proposition is a necessary and sufficient condition that μ to be a measure on $\mathcal{A}$. If moreover μ is finite then b) affirmation from above proposition represents also a necessary and sufficient condition that μ to be a measure on $\mathcal{A}$.

So, we consider the set of natural numbers $\mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and the function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$, $\mu(A) = \begin{cases} 0, & \text{if } A \text{ is finite} \\ +\infty, & \text{if } A \text{ is infinite.} \end{cases}$ Indeed the function μ is additive finite, has verified b) but is not a measure.

3 Lebesgue measure on $\mathbb{R}$

Let $\mathbb{R}$ be the set of real numbers. We denote by

$$\mathcal{S} = \{[a, b) \mid a \in \mathbb{R},\ b \in \mathbb{R}\}.$$

We denote by μ the function $\mu : \mathcal{S} \to \mathbb{R}_+$, defined by $\mu([a, b)) = b - a$.

Definition 7. It is called Lebesgue outer measure the function

$$\mu^* : \left\{ A \subset \mathbb{R} \mid (\exists)\ (E_n)_{n\in\mathbb{N}} \subset \mathcal{S},\ \bigcup_{n\in\mathbb{N}} E_n \supset A \right\} \to \mathbb{R}_+$$

defined by

$$\mu^*(A) = \inf \left\{ \sum_{n\in\mathbb{N}} \mu(E_n) \mid \bigcup_{n\in\mathbb{N}} E_n \supset A,\ (E_n)_{n\in\mathbb{N}} \in \mathcal{S} \right\}$$

Remark 2. Since $\mathbb{R} = \bigcup_n [-n, n) = \bigcup_{n\in\mathbb{Z}} [n, n+1)$ we have $\{A \subset \mathbb{R} \mid (\exists)\,(E_n)_{n\in\mathbb{N}} \subset \mathcal{S},\ \bigcup_n E_n \supset A\} = \mathcal{P}(\mathbb{R})$. Hence $\mu^* : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+$.

Theorem 1. The Lebesgue outer measure holds the following properties:
1. $\mu^*(\phi) = 0$
2. $A \subset B \subset \mathbb{R} \Rightarrow \mu^*(A) \leq \mu^*(B)$
3. $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}) \Rightarrow \mu^*(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$
4. $A \subset \mathbb{R}, t \in \mathbb{R} \Rightarrow \mu^*(A + t) = \mu^*(A)$
5. $\mu^* \mid \mathcal{S} = \mu$.

Definition 8. A subset $E \subset \mathbb{R}$ is called Lebesgue measurable if the following equality holds:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \mathcal{C}_E),\ \text{for any } A \in \mathcal{P}(\mathbb{R})$$

Remark 3. Since μ^* is increasing the equality is equivalent by inequality $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap CE)$. Hence equality is not trivial only for $A \in \mathcal{P}(\mathbb{R})$ with $\mu^*(A) < +\infty$.

We denote by $\mathcal{L} = \{E \in \mathcal{P}(\mathbb{R}) \mid E \text{ is Lebesgue measurable}\}$.

Theorem 2. For the couple $(\mathcal{L}, \mu^*)$ the following assertions hold:
1. $E, F \in \mathcal{L}$ we get $E \cup F \in \mathcal{L}$
2. $E, F \in \mathcal{L}$ we get $F \setminus E \in \mathcal{L}$
3. $(E_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ we get $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{L}$
4. $\mu^* \mid \mathcal{L}$ is a positive measure
5. $E \in \mathcal{P}(\mathbb{R})$, $\mu^*(E) = 0$, $F \subset E$ we get $F \in \mathcal{L}$ and $\mu^*(F) = 0$
6. $\mathcal{S} \subset \mathcal{L}$.
7. for any $E \in \mathcal{L}$ we get $\{x + t \mid x \in E\} = E + t \in \mathcal{L}$, for any $t \in \mathbb{R}$.

Conclusion 1. From above Theorem it follows that the Lebesgue measurable sets $\mathcal{L}$ form a σ-algebra which includes $\mathcal{S}$ and $\mu^* \mid_{\mathcal{L}}$ is a positive measure.

Remark 4. Restriction of μ^* to $\mathcal{L}$ is called induced measure by μ^* and was noted by $\overline{\mu}$.

Remark 5. The set $\mathcal{S}$ and the function μ have the following properties:
1. $E, F \in \mathcal{S} \Rightarrow E \cap F \in \mathcal{S}$
2. $E, F \in \mathcal{S} \Rightarrow E - F = \bigcup_{k=1}^{p} E_k$, $(E_k)_{1\leq k\leq p} \subset \mathcal{S}$, $E_i \cap E_j = \phi$.
3. $F \in \mathcal{S}$, $F = F_1 \cup F_2$, $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{S}$, $F_1 \cap F_2 = \phi$ it follows that $\mu(F) = \mu(F_1) + \mu(F_2)$.

4 Special properties of Lebesgue measure and measurability

We denote by μ restriction of outer Lebesgue measure to class of Lebesgue measurable sets and will call μ Lebesgue measure. We denote by $\mathcal{L}$ the sets Lebesgue measurable.

We denote by $\mathcal{B}$ the Borel sets on $\mathbb{R}$.

Any Borel set is Lebesgue measurable.

Lebesgue measure is only measure σ-finite on $\mathcal{B}$ whose restriction to $\mathcal{S}$ is the length intervals.

For any subset of $\mathbb{R}$ with outer Lebesgue measure finite there exists a Borel subset which contains and which has the same outer measure.

Any Lebesgue measurable set is reunion of a Borel set and a subset a Borel set of null Lebesgue measure.

For any Lebesgue measurable subset A of $\mathbb{R}$ with outer Lebesgue measure finite and for any $\varepsilon > 0$ there exists a finite reunion of intervals from $\mathcal{S}$ which differ to A whose outer Lebesgue measure is smaller than ε.

Lebesgue measure coincide with outer measures induced by restrictions of Lebesgue measure to $\mathcal{I}(\mathcal{S})$, $\mathcal{B}$, $\mathcal{L}$ we have

$$\begin{aligned} \mu^*(A) &= \inf\{\mu(E) \mid A \subset E, E \in \mathcal{B}\} \\ &= \inf\{\mu(M) \mid A \subset M, M \in \mathcal{L}\}. \end{aligned}$$

Theorem 3. Let $\mathcal{D}_{\mathbb{R}}$ be topology of $\mathbb{R}$, $\mathcal{F}_{\mathbb{R}}$ be the closed sets of $\mathbb{R}$,

$$\begin{aligned} &\mathcal{I}_1 = \{(a, b) \mid a, b \in \mathbb{R}\}; && \mathcal{I}_2 = \{(a, b] \mid a, b \in \mathbb{R}\}, \\ &\mathcal{I}_3 = \{[a, b] \mid a, b \in \mathbb{R}\}, && \mathcal{I}_4 = \{[a, +\infty) \mid a \in \mathbb{R}\}, \\ &\mathcal{I}_5 = \{(a, +\infty) \mid a \in \mathbb{R}\}, && \mathcal{I}_6 = \{(-\infty, a] \mid a \in \mathbb{R}\}, \\ &\mathcal{I}_7 = \{(-\infty, a) \mid a \in \mathbb{R}\}. \end{aligned}$$

Then $\mathcal{B} = \sigma(\mathcal{D}_{\mathbb{R}}) = \sigma(\mathcal{F}_{\mathbb{R}}) = \sigma(\mathcal{I}_k)$ for any k, $1 \leq k \leq 7$.

Corollary 1. $\mathbb{R}$ is a Borel set, $\{x\}$ is a Borel set for any $x \in \mathbb{R}$ and any subset at most countable of $\mathbb{R}$ is Borel set.

The outer Lebesgue measure is $\mu^*(A) = \inf\{\mu(D) \mid A \subset D, D \in \mathcal{D}_{\mathbb{R}}\}$, for any $A \subset \mathbb{R}$.

Any at most countable set of $\mathbb{R}$ has Lebesgue measure null.

5 Measurable functions

5.1 Simple functions

Proposition 2. Let X be a set. For every subset A of E we denote by 1_A the characteristic function of A (i.e. the function equal 1 on A and 0 on $X \setminus A$). The following assertions hold:

1. $\varphi_A = 0$ if and only if $A = \emptyset$; $\varphi_A = 1$ if and only if $A = X$.
2. $\varphi_A \leq \varphi_B$ if and only if $A \subseteq B$
3. $\varphi_A = \varphi_B$ if and only if $A = B$.
4. $\varphi_{A \cup B} = \varphi_A + \varphi_B - \varphi_A \cdot \varphi_B$
5. $\varphi_{A \cap B} = \varphi_A \cdot \varphi_B$
6. $\varphi_{A \setminus B} = \varphi_A (1 - \varphi_B)$
7. $\varphi_{A \cup B} = \varphi_A + \varphi_B$ if and only if $A \cap B = \emptyset$
8. $\varphi_{A \triangle B} = \varphi_A + \varphi_B - 2\varphi_A \cdot \varphi_B$.

Definition 9. Let $\mathcal{A}$ be a σ-algebra on X (i.e. $\mathcal{A}$ is a σ-ring and $X \in \mathcal{A}$). A function $f : X \to \mathbb{R}$ is called simple (i.e $\mathcal{A}$-simple) if $f = \sum_{i=1}^{n} c_i \varphi_{A_i}$ with $(c_i)_{i=\overline{1mn}} \subseteq \mathbb{R}$ and $(A_i)_{i=\overline{1,n}}$ a partition of X with sets of $\mathcal{A}$.

Remark 6. The condition as family $(A_i)_{i=\overline{1,n}}$ to be partition of X not is essential we can consider $(A_i)_{i=\overline{1,n}}$ an arbitrary family of X.

Examples. 1. The constant functions are simples. 2. The function sign, the function integer part on bounded interval and heaviside function are simple. 3. The Dirichlet function is simple ($f = 1 \cdot \varphi_Q + 0 \cdot \varphi_{\mathbb{R} \setminus Q}$).

Proposition 3. Let $\mathcal{A}$ be a σ-algebra on set X, $X \in \mathbb{R}$ and $f, g : X \to \mathbb{R}$ two simple functions. Then functions $f \pm g$, λf, $f \cdot g$, $|f|$, $\max\{f, g\}$, $\min\{f, g\}$ are simple.

Corollary 2. Let $f = \sum_{i=1}^{n} c_i \varphi_{A_i}$, $c_i \in \mathbb{R}$ and $A_i \in \mathcal{A}$ ($i = \overline{1,n}$). Then f is a simple function.

If X is a set, $\mathcal{A}$ is a σ-algebra on X and μ is a measure on $\mathcal{A}$ then $(X, \mathcal{A}, \mu)$ is called the space with measure.

Definition 10. Let $(X, \mathcal{A}, \mu)$ be a space with measure and P a propositional function defined on X i.e. for any $x \in X$, $P(x)$ is a proposition (true or false). We say that P is true almost everywhere (a.e.) if $P(x)$ is true for any $x \in X \setminus A$ with $A \subseteq X$ negligible (i.e. $\mu(A) = 0$).

Examples. 1. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f : X \to \mathbb{R}$ a function. We say that f is finite a.e. if there exists $A \subseteq X$ negligible such that f is finite on $X \setminus A$ i.e. $|f(x)| < +\infty$ for all $x \in X \setminus A$. 2. Let $(X, \mathcal{A}, \mu)$ be a space with measure, X being metrical space. A function $f : X \to \overline{\mathbb{R}}$ will say it is continuous a.e. if there exists $A \subseteq X$ negligible such that f is continuous on $X \setminus A$. If every $P_n (n \geq 1)$ is true a.e. then there exists $A \subseteq X$ negligible such that $P_n(x)$ is true for all $x \in X \setminus A$ and $n \geq 1$.

Definition 11. Let $(X, \mathcal{A})$ be a measurable space, Y a metrical space, τ_Y is the family of open sets of Y. We say that f is $\mathcal{A}$-measurable if $f^{-1}(\tau_Y) \subseteq \mathcal{A}$ i.e. $f^{-1}(G) \in \mathcal{A}$ for all $G \subseteq Y$, G open. If $X = \mathbb{R}$ and $\mathcal{A} = \mathcal{L}$ (resp. $\mathcal{A} = \mathcal{B}$) then f is called Lebesgue measurable (resp. Borel measurable). We say that f is measurable on $M \subseteq X$ (we can assume that $M \in \mathcal{A}$) if $M \cap f^{-1}(G) \in \mathcal{A}$, for all $G \in \tau_Y) \mid_M \subseteq \mathcal{A}$.

Examples. 1. The constant functions are measurable. 2. Let $X = [0, 1]$, $\mathcal{A} = \{\emptyset, [0, \frac{1}{2}), [\frac{1}{2}, 1], [0, 1]\}$ and $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$. Then f is not $\mathcal{A}$-measurable, since $G = (0, 1)$ is open and $f^{-1}(G) \notin \mathcal{A}$. 3. Let $A \subseteq \mathbb{R}$ a Lebesgue m. set but is not Borel m. i.e. $A \in \mathcal{L} \setminus \mathcal{B}$ and $f = 1_A$. Then f is Lebesgue m., but is not Borel m. Indeed for any $D \subset \mathbb{R}$, open we have $f^{-1}(G)$ equals with $A, CA, \mathbb{R}$, when $D \ni 1$, $D \not\ni 0$; $D \ni 0$, $D \not\ni 1$; $D \ni 0$, $D \ni 1$. Therefore $f^{-1}(D) \in \mathcal{L}$ and $f^{-1}(D) \notin \mathcal{B}$ if $D \ni 1$ and $D \not\ni 0$. etc.

Proposition 4. Let $(X, \mathcal{A})$ a measurable space, X, Y two metrical space, $f : X \to Y$ a measurable function and $g : Y \to Z$ a continuous function. Then $g \circ f$ is measurable.

Remark 7. If $f : \mathbb{R} \to \mathbb{R}$ is continuous any $g : \mathbb{R} \to \mathbb{R}$ Lebesgue measurable is not necessarily as $g \circ f$ Lebesgue measurable.

Theorem 4. Let $(X, \mathcal{A})$ a measurable space, Y a metrical space, τ_Y the open sets of Y and $f : X \to Y$ a function. Then the following assertions are equivalent:

1. f measurable i.e. $f^{-1}(\sigma_Y) \subseteq \mathcal{A}$.
2. $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{A}$, where $\mathcal{B}_Y = \sigma(\tau_Y)$ Borel sets of Y.
3. There exists $\mathcal{C} \subseteq 2^Y$ with $\sigma(\mathcal{C}) = \sigma(\tau_Y)$ (i.e. σ-algebra generated by $\mathcal{C}$ coincides with Borel sets of Y) such that $f^{-1}(\mathcal{C}) \subseteq \mathcal{A}$.

Proposition 5. Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ a function. Then the following assertions are equivalent.

1. f is measurable.
2. $\{x \in X \mid f(x) > \alpha\} \in \mathcal{A}$, for any $\alpha \in \mathbb{R}$.
3. $\{x \in X \mid f(x) \geq \alpha\} \in \mathcal{A}$, for any $\alpha \in \mathbb{R}$.
4. $\{x \in X \mid f(x) < \alpha\} \in \mathcal{A}$, for any $\alpha \in \mathbb{R}$.
5. $\{x \in X \mid f(x) \leq \alpha\} \in \mathcal{A}$, for any $\alpha \in \mathbb{R}$.

Corollary 3. Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ a function. Then the following assertions hold:

1. $\{x \in X \mid f(x) = \alpha\} \in \mathcal{A}$, for any $\alpha \in \mathbb{R}$;
2. $\{x \in X \mid \alpha < f(x) \leq \beta\} \in \mathcal{A}$, for any $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$;
3. $\{x \in X \mid \alpha \leq f(x) < \beta\} \in \mathcal{A}$, for $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ etc.

Corollary 4. Let $(X, \mathcal{A})$ a measurable space and $f : X \to \overline{\mathbb{R}}$ a function. Then f is measurable on set $A \in \mathcal{A}$ if and only if for any $\alpha \in \mathbb{R}$, $A \cap \{x \in X \mid f(x) > \alpha\} \in \mathcal{A}$.

Proposition 6. (The elementary properties of measurable function). Let $(X, \mathcal{A}, \mu)$ be a space with measurable and $f : X \to \overline{\mathbb{R}}$ an arbitrary function. Then the following assertions hold:

1. If f is constant then is measurable.
2. If f is measurable and $A \in \mathcal{A}$ it follows that f is measurable on A.
3. If there exists $(A_k)_{k \in K} \subseteq \mathcal{A}$ a family at most countable which cover X and f is measurable on A_k, for all $k \in K$ it follows that f is measurable.
4. If there exists $(A_k)_{k \in K} \subseteq \mathcal{A}$ a family at most countable which cover X and f is constant on A_k, for all $k \in K$ it follows that f is measurable.
5. If there exists $A \in \mathcal{A}$ with f constant on A and measurable on $X \setminus A$ it follows that f is measurable.
6. If μ is complete and $A \in \mathcal{A}$ is negligible results that f is measurable on A.
7. If μ is complete and f is measurable, changing values of f on a negligible set $A \in \mathcal{A}$, the obtained function $\tilde{f}$ is measurable.

Remark 8. The condition μ is complete from 6 and 7 is essential.

Example. Let $X = [0, 1]$, $\mathcal{A} = \{\emptyset, X\}$, $A = [0, \frac{1}{2}]$, $f = \varphi_A$ and $\mu : \mathcal{A} \to \mathbb{R}$, $\mu = 0$. Then $f = 0$ μ a.e. and is not measurable because $\{x \in X \mid f(x) > 0\} = A \notin \mathcal{A}$.

8. If f is measurable and $\tilde{f} : X \to \mathbb{R}$ is defined by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } |f(x)| < +\infty \\ 0, & \text{if } |f(x)| = +\infty \end{cases}$$

then $\tilde{f}$ is measurable.

9. If f is measurable and finite μ-a.e. it follows that the above function $\tilde{f}$ is measurable and $f = \tilde{f}$ μ-a.e.
10. If f is measurable then $\operatorname{sign}(f)$ is measurable.

Corollary 5. Let $(X, \mathcal{A})$ be a measurable space and 1_A the characteristic function of A, where $A \subseteq X$. Then 1_A measurable if and only if $A \in \mathcal{A}$.

Corollary 6. Let $(X, \mathcal{A})$ a measurable space $f, g : X \to \overline{\mathbb{R}}$ two measurable functions, $A \in \mathcal{A}$ and $h : X \to \overline{\mathbb{R}}$ a function defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in X \setminus A. \end{cases}$$

Then h is measurable.

Corollary 7. Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \overline{\mathbb{R}}$ a measurable function. Then for any $A \in \mathcal{A}$, a function $f \cdot \varphi_A$ is measurable.

Corollary 8. Let $(X, \mathcal{A}, \mu)$ be a space with measure, where X is metrical space and $\tau_X \subseteq \mathcal{A}$. Then a function $f : X \to \mathbb{R}$ is measurable if and only if f is measurable on each bounded and closed (resp. open) set of X.

Proposition 7. Let $(X, \mathcal{A})$ be a measurable space and $f : A \to \mathbb{R}$. Then f is simple if and only if f is measurable and takes a finite number of values.

Example. Let $M \subseteq \mathbb{R}$ a Lebesgue measurable set and $f : M \to \overline{\mathbb{R}}$ a continuous function μ-a.e. Then f is Lebesgue measurable.

Proof. Let A be the set of discontinuities of f. Then f is μ-negligible, hence $A \in \mathcal{L}$ and $M \setminus A \in \mathcal{L}$. Let $\alpha \in \mathbb{R}$. Obviously $\{x \in M \mid f(x) > \alpha\} = \{x \in M \setminus A \mid f(x) > \alpha\} \cup \{x \in A \mid f(x) > \alpha\} = M_1 \cup M_2$. Since f is continuous on $M \setminus A$ it follows that the set M_1 is open in $M \setminus A$, hence there exists $D \subseteq \mathbb{R}$ open with $D \cap (M \setminus A) = M_1$ and hence $M_1 \in \mathcal{L}$, hence D and $M \setminus A$ are in $\mathcal{L}$. We have $M_2 \subseteq A$ and $\lambda(A) = 0$, hence $M_2 \in \mathcal{L}$. Therefore $\{x \in M \mid f(x) > \alpha\} = M_1 \cup M_2 \in \mathcal{L}$ i.e. f is Lebesgue measurable. □

Proposition 8. Let $(X, \mathcal{A}, \mu)$ be a space with complete measure $f, g : X \to \overline{\mathbb{R}}$, $f = g$, μ-a.e. If f is measurable then g is measurable.

Remark 9. Condition μ complete is essential.

Example. Let $([0, 1], \mathcal{B}\mid_{[0,1]}, \mu)$ be the space with measure, $C \subseteq [0, 1]$ Cantor set and $A \subseteq \mathcal{C}$ a Lebesgue measurable set and is not Borel measurable and $f = \varphi_A$. Then $f = 0$ μ-a.e. (because $\mu(C) = 0$) and $A = \{x \in [0, 1] \mid f(x) > \frac{1}{2}\} \notin \mathcal{B} \mid |_{[0,1]}$, hence f is not $\mathcal{B}\mid_{[0,1]}$ measurable.

Theorem 5. Let $(X, \mathcal{A})$ be a measurable space. The following assertions hold:

1. If $f, g : X \to \mathbb{R}$ are measurable then functions $f \pm g$, λf, $|f|$, $\max\{f, g\}$, $\min\{f, g\}$, $f \cdot g$ are measurable.
2. If $f_n : X \to \overline{\mathbb{R}}$, $n \geq 1$ is a sequence of measurable functions then functions $\sup_{n\geq1} f_n$, $\inf_{n\geq1} f_n$, $\overline{\lim}_{n\to\infty} f_n$, $\underline{\lim}_{n\to\infty} f_n$ are measurable.
3. If $f, f_n : X \to \overline{\mathbb{R}}$ $(n \geq 1)$, where $f_n (n \geq 1)$ are measurable and $f_n \xrightarrow{s} f$, then f is measurable.

Corollary 9. Let $(X, \mathcal{A})$ be a measurable space and $f, g : X \to \overline{\mathbb{R}}$ two measurable functions. Then function $f + g$ is measurable.

Corollary 10. Let $(X, \mathcal{A})$ be a measurable space and $f : A \to \mathbb{R}$ a function. Then f is measurable if and only if $f^+ = \max\{f, 0\}$, $f^- = max\{-f, 0\}$ are measurable.

Since $f = f^+ - f^-$ the proof is obviously.

Theorem 6. Let $(X, \mathcal{A})$ a measurable space and $f : X \to \overline{\mathbb{R}}$ a measurable function. Then there exists a sequence of simple functions $f_n : X \to \mathbb{R}$, $(n \geq 1)$ such that $f_n \xrightarrow{s} f$. If f is bounded (resp. $f \geq 0$) the sequence $(f_n)_{n\geq 0}$ is uniform convergent (resp. increasing).

Proof. We suppose $f \geq 0$ and let $f_n : X \to \mathbb{R}$ $(n \geq 1)$ a sequence of functions defined by:

$$f_n(x) = \begin{cases} \frac{k-1}{2^n}, & \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \ (k = \overline{1, n \cdot 2^n}) \\ n, & f(x) \geq n \end{cases}$$

We put $A_0 = \{x \in X;\ f(x) \geq n\}$ and $A_k = \{x \in X \mid \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\}$, $k \in \overline{1, n - 2^n}$ where $n \geq 1$ is fixed. Obviously $(A_k)_{k=\overline{0,n\cdot 2^n}}$ is a measurable partition of X and we have $f_n(x) = \sum_{k=1}^{n\cdot 2^n} \frac{k-1}{2^n}\varphi_{A_k} + nA_0$, hence f_n is a simple function. It's clear that $f_{n+1} \geq f_n$ for all $n \geq 1$.

If $x \in X$ and there exists N natural with $f(x) \geq N$ it follows that $0 \leq f(x) - f_n(x) \leq \frac{1}{2^n}$, for all $n \geq N$, hence $f_n(x) \to f(x)$. Hence if $0 \leq f \leq N$ we deduce $f_n \xrightarrow{u} f$.

If $x \in X$ and $f(x) = \infty$, we have $f(x) \geq n$, for all $n \geq 1$, hence $f_n(x) = n$, for any $n \geq 1$ and hence $f_n(x) \to f(x)$.

Therefore in both cases we have $f_n(x) \to f(x)$, for all $x \in X$. We suppose now f is arbitrary measurable. Then f^+ and f^- are positive measurable and $f = f^+ - f^-$. There exists $f_n, g_n : X \to \mathbb{R}$ $(n \geq 1)$ two sequences of simple functions with $f_n \xrightarrow{s} f^+$ and $g_n \xrightarrow{s} f^-$, therefore $h_n = f_n - g_n$ $(n \geq 1)$ are simple and $h_n \xrightarrow{s} f$. □

Definition 12. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f_n : X \to \overline{\mathbb{R}}$ $(n \geq 1)$ a sequence of functions, finite μ-a.e. We say that sequence $(f_n)_{n\geq 1}$ converges μ-a.e. if there exists $A \subseteq X$ negligible and $f : X \to \overline{\mathbb{R}}$ such that the numerical sequence $(f_n(x))_{n\geq 1}$ is convergent (in $\mathbb{R}$) and has limit $f(x)$ for all $x \in X \setminus A$. We write now $f_n \xrightarrow{\mu-a.e} f$.

Remark 10. 1. If $(f_n)_{n\geq 1}$ is a sequence of functions finite μ-a.e. and $f_n \xrightarrow{\mu-a.e} f$ then f is finite μ-a.e.

2. If sequence $(f_n)_{n\geq 1}$ converges μ-a.e. then function limit is uniquely determinated μ-a.e., i.e. if $f_n \xrightarrow{\mu-a.e} f$ and $f_n \xrightarrow{\mu-a.e} g$ then $f = g$ μ-a.e.

3. If $(f_n)_{n\geq 1}$ is a sequence of functions finite μ-a.e. then there exists $A \subseteq X$ negligible such that each f_n $(n \geq 1)$ is finite on $X \setminus A$.

Proposition 9. Let $(X, \mathcal{A}, \mu)$ be a space with complete measure and $f_n : X \to \mathbb{R}$ $(n \geq 1)$ a sequence of measurable, finite μ-a.e. functions which converges μ-a.e. Then function limit f is measurable.

Remark 11. Condition μ complete is essential.

Example. Let $X = [0, 1]$, $\mathcal{A} = \{\emptyset, X\}$, $\mu = 0$, $f = \varphi_{\{0\}}$ and $f_n = f$, for all $n \geq 1$. Then $f_n \xrightarrow{\mu-a.e} f$, because $\mu(X) = 0$, but f is not measurable.

Definition 13. Let $(X, \mathcal{A}, \mu)$ be a space with measure and $f_n : X \to \overline{\mathbb{R}}$ $(n \geq 1)$ a sequence of measurable functions, f μ-a.e finite. We say that $(f_n)_{n\geq 1}$ converges almost uniformly (a.u.) if there exists $f : X \to \overline{\mathbb{R}}$ measurable such that for any $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{A}$ with $\mu(A_\varepsilon) < \varepsilon$ and $f_n \xrightarrow[X\setminus A_\varepsilon]{u} f$. We write then $f_n \xrightarrow{a.u} f$.

Remark 12. If $(f_n)_{n\geq 1}$ converges a.u. to f not necessarily it results that $(f_n)_{n\geq 1}$ converges uniformly μ-a.e. (i.e. a complementary of a negligible set) to f.

Example. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \geq 1)$, $f_n = \varphi_{A_n}$, where $A_n = \left(\frac{1}{n}, \frac{2}{n}\right)$. Then $f_n \xrightarrow{a.u} 0$ and $f_n \xrightarrow{\mu-a.e} 0$.

Proposition 10. Let $(X, \mathcal{A}, \mu)$ be a space with measure, $f_n : X \to \overline{\mathbb{R}}$ $(n \geq 1)$ a sequence of measurable, finite μ-a.e. functions and $f : X \to \overline{\mathbb{R}}$ a measurable function. If $f_n \xrightarrow{a.u} f$, then f is finite μ-a.e and $f_n \xrightarrow{\mu-a.e} f$.

Theorem 7. (Egorov) Let $(X, \mathcal{A}, \mu)$ be a space with finite measure and $f, f_n : X \to \overline{\mathbb{R}}$ $(n \geq 1)$ measurables, finite μ-a.e such that $f_n \xrightarrow{\mu-a.e} f$. Then $f_n \xrightarrow{a.u} f$.

Remark 13. If μ is not finite then Theorem is not true.

Example. Let $(\mathbb{R}, \mathcal{L}, \mu)$ be a space with measure and $f_n : X \to \mathbb{R}$ $(n \geq 1)$ $f_n = \varphi_{A_n}$, where $A_n = [n, n+1]$. Then $f_n \xrightarrow{s} 0$ (hence $f_n \xrightarrow{\mu-a.e} 0$), but $f_n \stackrel{a.u}{\nrightarrow} 0$.

Lemma 1. Let $E \subseteq \mathbb{R}$ a Lebesgue measurable set and $f : E \to \mathbb{R}$ a Lebesgue measurable function. Then $(\forall)\ \varepsilon > 0$, $(\exists) A_\varepsilon \subseteq E$ closed such that $\mu(A \setminus A_\varepsilon) < \varepsilon$ and $f\ |_{A_\varepsilon}$ continuous.

Lemma 2. Let $E \subseteq \mathbb{R}$ a Lebesgue measurable set with $\mu(E) < +\infty$ and $f : E \to \overline{\mathbb{R}}$ a Lebesgue measurable function finite μ-a.e. Then $(\forall)\ \varepsilon > 0$, $(\exists) A_\varepsilon \subseteq E$ closed such that $\mu(E \setminus A_\varepsilon) < \varepsilon$ and $f\ |_{A_\varepsilon}$ continuous.

Theorem 8. (Luzin) Let $E \subseteq \mathbb{R}$ a Lebesgue measurable set and $f : E \to \overline{\mathbb{R}}$ a function Lebesgue measurable, finite μ-a.e. Then $(\forall)\ \varepsilon > 0$, $(\exists) A_\varepsilon \subseteq E$ closed with $\mu(A \setminus A_\varepsilon) < \varepsilon$ and $f\ |_{A_\varepsilon}$ continuous.

Corollary 11. Let $E \subseteq \mathbb{R}$ a Lebesgue measurable set and $f : E \to \overline{\mathbb{R}}$ a function finite μ-a.e. Then f is Lebesgue measurable and if and only if f is almost continuous (i.e. $\forall$ $\varepsilon > 0$, $(\exists) A_\varepsilon \subseteq X$ closed with $\mu(X \setminus A_\varepsilon) < \varepsilon$ and $f\ |_{A_\varepsilon}$ continuous.

Remark 14. To understand this Theorem we observe that there exists functions $f : \mathbb{R} \to \mathbb{R}$ Lebesgue m., but which are discontinuous in every point (The Dirichlet function).

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