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Homomorphic embeddings in n-groups

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ABSTRACT

We prove that an cancellative n-groupoid $\mathcal{A}$ can be homotopic embedded in an n-group if and only if in $\mathcal{A}$ are satisfied all n-ary Malcev conditions. Now we shall prove that in the presence of associative law we obtain homomorphic embeddings. Furthermore, if $\mathcal{A}$ has a lateral identity a such embeddings is assured by a subset of n-ary Malcev conditions - unary Malcev conditions.
Keywords: cancellation law, covering semigroup, homotopic embedding, n-ary Malcev conditions, n-groupoid, unary Malcev conditions.

We prove that an cancellative n-groupoid $\mathcal{A}$ can be homotopic embedded in an n-group if and only if in $\mathcal{A}$ are satisfied all n-ary Malcev conditions.

Now we shall prove that in the presence of associative law we obtain homomorphic embeddings. Furthermore, if $\mathcal{A}$ has a lateral identity a such embeddings is assured by a subset of n-ary Malcev conditions - unary Malcev conditions.

For an abbreviation we shall use the follwing notations(see [1]):

$$(x_1, x_2, \ldots, x_n) = x_1^n,$$

respectively x^n if

$$x_1 = x_2 = \cdots = x_n = x.$$

Let $\mathcal{A} = (A, \alpha)$ be an n-groupoid (i.e $\alpha : A^n \to A$). If α satisfies the associative law

$$\alpha(\alpha(x_1^n), x_{n+1}^{2n-1}) = \alpha(x_1^i, \alpha(x_{i+1}^{n+i}), x_{n+i+1}^{2n-1})$$

for $i = 1, 2, \ldots, n-1$ and for all $x_1, \ldots, x_{2n-1}$ in A then $\mathcal{A}$ is an **n-semigroup**.

The sequence a_1^{n-1} is an lateral identity in the n-groupoid $\mathcal{A}$ if

$$\alpha(a_1^{n-1}, x) = \alpha(x, a_1^{n-1}) = x$$

for all x in A.

The following laws, wich may of may not hold in a given n-groupoid $\mathcal{A}$, are known as **left** and **right cancellation laws**, respectively,

$$\alpha(u_1^{n-1}, x) = \alpha(u_1^{n-1}, y) \Rightarrow x = y$$

$$\alpha(x, u_1^{n-1}) = \alpha(y, u_1^{n-1}) \Rightarrow x = y$$

An n-groupoid $\mathcal{A}$ is a **cancellation n-groupoid** if

$$\alpha(u_1^{i-1}, x, u_{i+1}^n) = \alpha(u_1^{i-1}, y, u_{i+1}^n) \Rightarrow x = y$$

for $i = 1, 2, \ldots, n-1$.

In [5] was proved that an n-semigroup wich is left and right cancellative is a cancellation n-semigroup.

An important concept in the theory of n-semigroups is that of a covering semigroup.

Definition 1. *(see [5]) A binary $\overline{\mathcal{A}} = (\overline{A}, \cdot)$ semigroup is said to be a* ***covering semigroup*** *of an n-semigroup $\mathcal{A} = (A, \alpha)$ provided $\overline{\mathcal{A}}$ has the following properties:*

- *the set A is a generating subset of $\overline{\mathcal{A}}$;*
- *$\alpha(a_1^n) = a_1 \cdot a_2 \ldots a_n$ for all $a_1, \ldots, a_n \in A$.*

Generalizing an result from [5] we have

Theorem 1. *Every cancellation n-semigroup has a cancellation covering semigroup.*

Outline of proof. Let $\mathcal{A} = (A, \alpha)$ be an cancellation n-semigroup. Denote by $S' = (S', \cdot)$ the free semigroup with identity generated by the set A. Let us consider the binary relation $\pi \subseteq S'^2$ defined by: $s \pi s'$ iff

1. *there exist $s_1, s_2, s_3 \in S'$ such that $\lambda(s_2) = n$ (where $\lambda(s_2)$ is the lenght of s_2), $s = s_1 s_2 s_3$ and $s' = s_1 \alpha(s_2) s_3$,or*
2. *$\lambda(s) = \lambda(s') < n$ and there is a $s'' \in S'$ with $\lambda(s'') = n - \lambda(s)$ such that $\alpha(ss') = \alpha(ss'')$, or*

3. $s = 1$ *(the identity of* $\mathcal{S}'$*),* $\lambda(s') = n-1$ *and* $\alpha(s', a) = a$ *for some* $a \in A$.

Denote by ρ the equivalence on S' generated by π. Then ρ is a congruence on $\mathcal{S}'$ and $\mathcal{S}'/\rho$ is a cancellation covering semigroup of $\mathcal{A}$.

It is easy to prove the following

Lemma 1. *Let be* $\overline{\mathcal{A}}$ *a covering semigroup of the* n*-semigroup* $\mathcal{A}$*. If* $\overline{\mathcal{A}}$ *can be homomorphic embedded in a group then* $\mathcal{A}$ *can be homomorphic embedded in a* n*-group.*

Theorem 2. *A cancellation* n*-semigroup* $\mathcal{A} = (A, \alpha)$ *without lateral identities can be homorphical embedded in a* n*-group iff in* $\mathcal{A}$ *are satisfied all* n*-ary Malcev conditions.*

Proof. Suppose that $\mathcal{A}$ can be homomorphical embedded in an n-group $\mathcal{G}$. All n-ary Malcev conditions are satisfied in $\mathcal{G}$. Consequently, these conditions are satisfied in $\mathcal{A}$.

Conversely, assume that all n-ary Malcev conditions are satisfied in $\mathcal{A}$. By Lemma 1 it is sufficient to prove that the covering semigroup $\mathcal{S}'(\mathcal{A})/\rho$ is homomorphic embeddable in a binary group. $\mathcal{A}$ being without lateral identities, [1] is a prime unit in $\mathcal{S}'(\mathcal{A})/\rho$. Therefore it is sufficient to prove that the semigroup $\mathcal{S}(\mathcal{A})/\rho = (\mathcal{S}'(\mathcal{A})/\rho - \{[1]\}, \cdot)$ is embeddable in a group. There exists such an embedding iff in $\mathcal{S}(\mathcal{A})/\rho$ are satisfied all binary Malcev conditions(see [3]). Since $\{[a] \mid a \in A\}$ is a generating set of $\mathcal{S}(\mathcal{A})/\rho$ it is sufficient (see [3]) to consider only Malcev conditions according the table

L_i	$\bar{L}_i$	R_i	$\bar{R}_i$
$[a_i][s_i]$	$[u_i][\bar{s}_i]$	$[w_i][\bar{a}_i]$	$[\bar{w}_i][t_i]$
$[u_i][s_i]$	$[a_i][\bar{s}_i]$	$[w_i][t_i]$	$[\bar{w}_i][\bar{a}_i]$

(1)

Let I be a Malcev sequence and $\sigma(I)$ the corresponding system of equalities. Adjoining the closing equality to $\sigma(I)$ we obtain the system $\widetilde{\sigma}(I)$. To each equality of $\widetilde{\sigma}(I)$ we assign a tag - the corresponding pair of symbols of I.

Example. Let $I = R_1L_1R_2L_2R_3\overline{L}_2\overline{R}_3\overline{R}_2\overline{L}_1\overline{R}_1$. The tagged system $\widetilde{\sigma}(I)$ is

(R_1L_1) $[w_1][\bar{a}_1] = [u_1][s_1]$
(L_1R_2) $[a_1][s_1] = [w_2][t_2]$
(R_2L_2) $[w_2][\bar{a}_2] = [u_2][s_2]$
(L_2R_3) $[a_2][s_2] = [w_3][t_3]$
$(R_3\overline{L}_2)$ $[w_3][\bar{a}_3] = [a_2][\bar{s}_2]$
$(\overline{L}_2, \overline{R}_3)$ $[u_2][\bar{s}_2] = [\bar{w}_3][\bar{a}_3]$
$(\overline{R}_3, \overline{R}_2)$ $[\bar{w}_3][t_3] = [\bar{w}_2][\bar{a}_2]$
$(\overline{R}_2, \overline{L}_1)$ $[\bar{w}_2][t_2] = [a_1][\bar{s}_1]$
$(\overline{L}_1, \overline{R}_1)$ $[u_1][\bar{s}_1] = [\bar{w}_1][\bar{a}_1]$
$(\overline{R}_1, R_1)$ $[\bar{w}_1][t_1] = [w_1][t_1]$ (the closing equality)

From the definition of the congruence relation ρ it folows:

- if $[x] = [y]$ then $\lambda(x) \equiv \lambda(y)(\text{mod } n-1)$, where $\lambda(x)$ is the length of x;
- in each class $[x]$ there is an element x' with $\lambda(x') \leq n-1$.

Consequently, we can suppose that in the table 1 each representative has the length $\leq n-1$.

Now we construct a new system of equalities $\widetilde{\sigma}$ in wich member has the length $\equiv 1(\text{mod } n-1)$. Let a be an element of A.

1. If L_1 is the first symbol of I,

$$(L_1-)\ [a_1][s_1] = [x][y]$$

we choose $0 \leq j_1 \leq n-1$ such that $\lambda(s_1) + j_1 \equiv 0$.

2. If R_1 is the first symbol of I,

$$(R_1-)\ [w_1][\overline{a_1}] = [x][y]$$

we choose $0 \leq i_1 \leq n-1$ such that $i_1 + \lambda(w_1) \equiv 0$.

We obtain the first equality of $\widetilde{\sigma}(I)$ by multiplying the first equality of $\overline{\sigma}(I)$ on the right by a^{j_1} in the first case and on the left by a^{i_1} in the second case.

We obtain the second equality of $\widetilde{\sigma}(I)$ from the second equality of $\overline{\sigma}(I)$ in the following way: if the first (second) factor of the left member of the second equality of $\overline{\sigma}(I)$ is equal to the first (second) factor of the right member in the first equality of $\overline{\sigma}(I)$ then we multiply the second equality of $\overline{\sigma}(I)$ by the left by a^{i_1} and by the right by a^{j_2}(respectively, by the left by a^{i_2} and by the right by a^{j_1}) where $0 \leq i_2, j_2 \leq n-1$ are such that the length of the left member of new equality be $\equiv 1$.

In the same manner we obtain the $\underline{k}th$ equality of $\widetilde{\sigma}(I)$ from the $\underline{k}th$ equality of $\overline{\sigma}(I)$.

Example. We apply this procedure to the system $\overline{\sigma}(I)$ considered in the previous example.

Suppose $n = 5, \lambda(u_1) = 2, \lambda(s_1) = 3, \lambda(\bar{s}_1) = 3, \lambda(u_2) = 2, \lambda(s_2) = 1, \lambda(\bar{s}_2) = 1, \lambda(w_1) = 4, \lambda(\bar{w}_1) = 4, \lambda(t_1) = 2, \lambda(w_2) = 2, \lambda(\bar{w}_2) = 2, \lambda(t_2) = 2, \lambda(w_3) = 1, \lambda(\bar{w}_3) = 2, \lambda(t_3) = 1$.

The tagged system $\widetilde{\sigma}(I)$ is

(R_1L_1) $w_1\bar{a}_1 \equiv u_1s_1$
(L_1R_2) $aa_1s_1 \equiv aw_2t_2$
(R_2L_2) $aw_2\bar{a}_2a \equiv au_2s_2a$
(L_2R_3) $a^2a_2s_2a \equiv a^2w_3t_3a$
$(R_3\overline{L}_2)$ $a^2w_3\bar{a}_3a \equiv a^2a_2\bar{s}_2a$
$(\overline{L}_2\overline{R}_3)$ $au_2\bar{s}_2a \equiv aw_3a_3a$
$(\overline{R}_3\overline{R}_2)$ $aw_3t_3a \equiv a\bar{w}_2\bar{a}_2a$
$(\overline{R}_2\overline{L}_1)$ $a\bar{w}_2t_2 \equiv aa_1\bar{s}_1$
$(\overline{L}_1\overline{R}_1)$ $u_1\bar{s}_1 \equiv \bar{w}_1\bar{a}_1$
$(\overline{R}_1R_1)$ $\bar{w}_1t_1a^3 \equiv w_1t_1a^3$

Now we prove that $\widetilde{\sigma}(I)$ is a system of equalities corresponding to same Malcev sequence I. Hence, we must show that the equalities of $\widetilde{\sigma}(I)$ are obtained according the table

L_k	$\bar{L}_k$	R_k	$\bar{R}_k$
$(a^{i_k}a_k)(s_k a^{j_k})$	$(a^{i'_k}u_k)(\bar{s}_k a^{j'_k})$	$(a^{i_k}w_k)(\bar{a}_k a^{j_k})$	$(a^{i'_k}\bar{w}_k)(t_k a^{j'_k})$
$(a^{i'_k}u_k)(s_k a^{j_k})$	$(a^{i_k}a_k)(\bar{s}_k a^{j'_k})$	$(a^{i_k}w_k)(t_k a^{j'_k})$	$(a^{i'_k}\bar{w}_k)(\bar{a}_k a^{j_k})$

(2)

Let be L_q any symbol of I. We use an inductive argument on $n(L_q)$ = the number of L symbols between L_q and $\overline{L}_q$. Suppose $n(L_q) = 0$. Then $q = 1$. We have two cases.

Case 1. L_1 is the first symbol of I. Then

$$(L_1-)\ a_1 s_1 a^{j_1} \equiv x_1 y_1 a^{j_1}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-\overline{L}_1)\ x_2 y_2 a^{j_2} \equiv a_1 \bar{s}_1 a^{j_2}$$
$$(\overline{L}_1-)\ a^{i_1} u_1 \bar{s}_1 a^{j_2} \equiv a^{i_1} x_3 y_3 a^{j_2}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-L_1)\ a^{i_1} x_4 y_4 a^{j_3} \equiv a^{i_1} u_1 s_1 a^{j_3}$$

We have that $\lambda(s_1)+j_1 \equiv 0$, $\lambda(\bar{s}_1)+j_2 \equiv 0$, $i_1+\lambda(u_1)+\lambda(\bar{s}_1)+j_2 \equiv 1$ and $i_1+\lambda(u_1)+\lambda(s_1)+j_3 \equiv 1$. Hence $i_1+\lambda(u_1) \equiv 1$ and then $\lambda(s_1)+j_3 \equiv 0$ implies $j_3 = j_1$ and

L_1	$\bar{L}_1$
$(a_1)(s_1 a^{j_1})$	$(a^{i_1}u_1)(\bar{s}_1 a^{j_2})$
$(a^{i_1}u_1)(s_1 a^{j_1})$	$(a_1)(\bar{s}_1 a^{j_2})$

(3)

Case 2. L_q is not the first symbol of I. Then

$$(\,)\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-L_q)\ a^{i_1} x_1 y_1 a^{j_1} \equiv a^{i_1} u_q s_q a^{j_1}$$
$$(\overline{L}_q-)\ a^{i_2} a_q s_q a^{j_1} \equiv a^{i_2} x_2 y_2 a^{j_1}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-\overline{L}_q)\ a^{i_2} x_3 y_3 a^{j_2} \equiv a^{i_2} a_q \bar{s}_q a^{j_2}$$
$$(\overline{L}_q)\ a^{i_3} u_q \bar{s}_q a^{j_2} \equiv a^{i_3} x_4 y_4 a^{j_2}$$

We have that $i_1+\lambda(u_q)+\lambda(s_q)+j_1 \equiv 1$, $i_2+1 = i_1+\lambda(u_q)$, $i_2+1+\lambda(\bar{s}_q)+j_2 \equiv 1$ and $i_3+\lambda(u_q)+\lambda(\bar{s}_q)+j_2 \equiv 1$. Hence $i_3+\lambda(u_q) = i_2+1 = i_1+\lambda(u_q)$ and thus $i_3 = i_1$, and then

L_q	$\bar{L}_q$
$(a^{i_2}a_q)(s_q a^{j_1})$	$(a^{i_1}u_q)(\bar{s}_q a^{j_2})$
$(a^{i_1}u_q)(s_q a^{j_1})$	$(a^{i_2}a_q)(\bar{s}_q a^{j_2})$

(4)

Suppose now that this results is true for all $n(L) < d$ and $n(L_q) = d$. Then between L_q and $\overline{L}_q$ there exists the symbols $L_{q+1},\dots,L_{q+d}$ and $\overline{L}_{q+1},\dots,\overline{L}_{q+d}$. Again we have two cases.

Case1. L_q is the first symbol of I. Then $q = 1$.

$$(L_1-)\ a_1 s_1 a^{j_1} \equiv x_1 y_1 a^{j_1}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-L_2)\ x_2 y_2 a^{j_2} \equiv u_2 s_2 a^{j_2}$$
$$(L_2-)\ a^{i_2} a_2 s_2 a^{j_2} \equiv a^{i_2} x_3 y_3 a^{j_2}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-\overline{L}_2)\ a^{i_3} x_4 y_4 a^{j_3} \equiv a^{i_3} a_2 \bar{s}_2 a^{j_3}$$
$$(\overline{L}_2-)\ a^{i_4} u_2 \bar{s}_2 a^{j_3} \equiv a^{i_4} x_5 y_5 a^{j_3}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-\overline{L}_1)\ a^{i_4} x_6 y_6 a^{j_4} \equiv a^{i_4} a_1 \bar{s}_1 a^{j_4}$$
$$(\overline{L}_1-)\ a^{i_5} u_1 \bar{s}_1 a^{j_4} \equiv a^{i_5} x_7 y_7 a^{j_4}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(-L_1)\ a^{i_5} x_8 y_8 a^{j_5} \equiv a^{i_5} u_1 s_1 a^{j_5}$$

We have

$$\lambda(s_1)+j_1 \equiv 0$$
$$\lambda(u_2)+\lambda(s_2)+j_2 \equiv 1$$
$$i_2+1+\lambda(s_2)+j_2 \equiv 1$$
$$i_3+1+\lambda(\bar{s}_2)+j_3 \equiv 1$$
$$i_4+\lambda(u_2)+\lambda(\bar{s}_2)+j_3 \equiv 1$$
$$i_4+1+\lambda(\bar{s}_1)+j_4 \equiv 1$$
$$i_5+\lambda(u_1)+\lambda(\bar{s}_1)+j_4 \equiv 1$$
$$i_5+\lambda(u_1)+\lambda(s_1)+j_5 \equiv 1$$

Since $n(L_2) < d$, from

L_2	$\bar{L}_2$
$(a^{i_2}a_2)(s_2 a^{j_2})$	$(a^{i_4}u_2)(\bar{s}_2 a^{j_3})$
$(u_2)(s_2 a^{j_2})$	$(a^{i_3}a_q 2)(\bar{s}_2 a^{j_3})$

(5)

it follows that $i_2 = i_3$ and $i_4 = 0$. Now from $i_4+1+\lambda(\bar{s}_1)+j_4 \equiv 1$ it follows $\lambda(\bar{s}_1)+j_4 \equiv 0$, and from $i_5+\lambda(u_1)+\lambda(\bar{s}_1)+j_4 \equiv 1$ we obtain $i_5+\lambda(u_1) \equiv 1$. Now $i_5+\lambda(u_1)+\lambda(s_1)+j_5 \equiv 1$ implies $\lambda(s_1)+j_5 \equiv 0$. From the first equality we obtain $\lambda(s_1)+j_1 \equiv 0$. Therefore, $j_5 = j_1$ and we have

L_1	$\bar{L}_1$
$(a_1)(s_1 a^{j_1})$	$(a^{i_5}u_1)(\bar{s}_1 a^{j_4})$
$(a^{i_5}u_1)(s_1 a^{j_1})$	$(a_1)(\bar{s}_1 a^{j_4})$

(6)

Case 2. L_q is not the first symbol of I. Then

$$
\begin{aligned}
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(-L_q)\ a^{i_q}x_1y_1a^{j_q} \equiv a^{i_q}u_qs_qa^{j_q}\\
&(L_q-)\ a^{i'_q}a_qs_qa^{j_q} \equiv a^{i'_q}x_2y_2a^{j_q}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(-L_{q+1})\ a^{i'_q}x_3y_3a^{j_{q+1}} \equiv a^{i'_q}u_{q+1}s_{q+1}a^{j_{q+1}}\\
&(L_{q+1}-)\ a^{i_{q+1}}a_{q+1}s_{q+1}a^{j_{q+1}} \equiv a^{i_{q+1}}x_4y_4a^{j_{q+1}}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(-\overline{L}_{q+1})\ a^{i'_{q+1}}x_5y_5a^{j'_{q+1}} \equiv a^{i'_{q+1}}a_{q+1}\bar{s}_{q+1}a^{j'_{q+1}}\\
&(\overline{L}_{q+1}-)\ a^{i''_{q+1}}u_{q+1}\bar{s}_{q+1}a^{j'_{q+1}} \equiv a^{i''_{q+1}}x_6y_6a^{j'_{q+1}}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(-\overline{L}_q)\ a^{i''_{q+1}}x_7y_7a^{j''_{q+1}} \equiv a^{i''_{q+1}}a_q\bar{s}_qa^{j''_{q+1}}\\
&(\overline{L}_q-)\ a^{i'''_{q+1}}u_q\bar{s}_qa^{j''_{q+1}} \equiv a^{i'''_{q+1}}x_8y_8a^{j''_{q+1}}
\end{aligned}
$$

We have

$$
\begin{aligned}
&i_q+\lambda(u_q)+\lambda(s_q)+j_q \equiv 1\\
&i'_q+1+\lambda(s_q)+j_q \equiv 1\\
&i'_q+\lambda(u_{q+1})+\lambda(s_{q+1})+j_{q+1} \equiv 1\\
&i_{q+1}+1+\lambda(s_{q+1})+j_{q+1} \equiv 1\\
&i'_{q+1}+1+\lambda(\bar{s}_{q+1})+j'_{q+1} \equiv 1\\
&i''_{q+1}+\lambda(u_{q+1})+\lambda(\bar{s}_{q+1})+j'_{q+1} \equiv 1\\
&i''_{q+1}+1+\lambda(\bar{s}_q)+j''_{q+1} \equiv 1\\
&i'''_{q+1}+\lambda(u_q)+\lambda(\bar{s}_q)+j''_{q+1} \equiv 1
\end{aligned}
$$

Since $n(L_{q+1}) = d-1$ from

L_{q+1}	$\bar{L}_{q+1}$
$(a^{i_{q+1}}a_{q+1})(s_{q+1}a^{j_{q+1}})$	$(a^{i''_{q+1}}u_{q+1})(\bar{s}_{q+1}a^{j'_{q+1}})$
$(a^{i'_q}u_{q+1})(s_{q+1}a^{j_{q+1}})$	$(a^{i'_{q+1}}a_{q+1})(\bar{s}_{q+1}a^{j'_{q+1}})$

(7)

it follows that $i_{q+1} = i'_{q+1}$ and $i'_q = i''_{q+1}$.

Now from $i'_q+1+\lambda(s_q)+j_q \equiv 1$ and $i''_{q+1}+1+\lambda(\bar{s}_q)+j''_{q+1} \equiv 1$ we get $\lambda(s_q)+j_q \equiv \lambda(\bar{s}_q)+j''_{q+1}$.

From $i'''_{q+1}+\lambda(u_q)+\lambda(\bar{s}_q)+j''_{q+1} \equiv 1$, $i'''_{q+1}+\lambda(u_q)+\lambda(s_q)+j_q \equiv 1$ and $i_q+\lambda(u_q)+\lambda(s_q)+j_q \equiv 1$ it follows that $i'''_{q+1}+\lambda(u_q) \equiv i_q+\lambda(u_q)$, therefore $i_q \equiv i'''_{q+1}$, and we have

L_q	$\bar{L}_q$
$(a^{i'_q}a_q)(s_qa^{j_q})$	$(a^{i_q}u_q)(\bar{s}_qa^{j''_{q+1}})$
$(a^{i'_q}u_q)(s_qa^{j_q})$	$(a^{i'_q}a_q)(\bar{s}_qa^{j''_{q+1}})$

(8)

Similar arguments for R symbols complete the proof.

Example The corresponding table 2 for $\widetilde{\sigma}(I)$ considered above is

L_1	$\bar{L}_1$	R_1	$\bar{R}_1$
$(aa_1)s_1$	$u_1\bar{s}_1$	$w_1\bar{a}_1$	$\bar{w}_1(t_1a^3)$
u_1s_1	$(aa_1)\bar{s}_1$	$w_1(t_1a^3)$	$\bar{w}_1\bar{a}_1$

R_2	$\bar{R}_2$	R_3	$\bar{R}_3$
$(aw_2)(\bar{a}_2a)$	$(a\bar{w}_2)t_2$	$(a^2w_3)(\bar{a}_3a)$	$(a\bar{w}_3)(t_3a)$
$(aw_2)t_2$	$(a\bar{w}_2)(\bar{a}_2a)$	$(a^2w_3)(t_3a)$	$(a\bar{w}_3)(\bar{a}_3a)$

(9)

All elements of table 2 are long products. It is easy to see that they have length n or $2n-1$. From the definition of the congruence relation ρ it follows that if $x \equiv y(\text{mod } \rho)$ and $\lambda(x), \lambda(y) \equiv 1$, then $\alpha(x) = \alpha(y)$, where $\alpha(x), \alpha(y)$ are the corresponding long products.

It is easy to prove that in terms of $\mathcal{A}$ the system $\widetilde{\sigma}(I)$ is a system of equalities corresponding to the same Malcev sequence I in wich appears now n-ary symbols.

For example, let be

L_k	$\bar{L}_k$
$(a^{i_k}a_k)(s_ka^{j_k})$	$(a^{i'_k}u_k)(\bar{s}_ka^{j'_k})$
$(a^{i'_k}u_k)(s_ka^{j_k})$	$(a^{i_k}a_k)(\bar{s}_ka^{j'_k})$

(10)

Case 1. Suppose

$$
\begin{aligned}
&i_k+1+\lambda(s_k)+j_k = n\\
&i'_k+\lambda(u_k)+\lambda(s_k)+j_k = n
\end{aligned}
$$

Then

$$
\begin{aligned}
\alpha((a^{i_k}a_k)(s_ka^{j_k})) &= \alpha(a^{i_k}, a_k, s_k, a^{j_k})\\
\alpha((a^{i'_k}u_k)(s_ka^{j_k})) &= \alpha(a^{i'_k}, u_k, s_k, a^{j_k})
\end{aligned}
$$

Now if

$$
\begin{aligned}
&i'_k+\lambda(u_k)+\lambda(\bar{s}_k)+j'_k = n\\
&i_k+1+\lambda(\bar{s}_k)+j'_k = n
\end{aligned}
$$

we have

$$
\begin{aligned}
\alpha((a^{i'_k}u_k)(\bar{s}_ka^{j'_k})) &= \alpha(a^{i'_k}, u_k, \bar{s}_k, a^{j'_k})\\
\alpha((a^{i_k}a_k)(\bar{s}_ka^{j'_k})) &= \alpha(a^{i_k}, a_k, \bar{s}_k, a^{j'_k})
\end{aligned}
$$

and we obtain the table

$L_k^{i_{k+1}}$	$\bar{L}_k^{i_{k+1}}$
$\alpha(a^{i_k}, a_k, s_k, a^{j_k})$	$\alpha(a^{i_k}, u_k, \bar{s}_k, a^{j'_k})$
$\alpha(a^{i'_k}, u_k, s_ka^{j_k})$	$\alpha(a^{i_k}, a_k, \bar{s}_k, a^{j'_k})$

(11)

Suppose now that

$$
\begin{aligned}
&i'_k+\lambda(u_k)+\lambda(\bar{s}_k)+j'_k = 2n-1\\
&i_k+1+\lambda(\bar{s}_k)+j'_k = 2n-1
\end{aligned}
$$

Then

$$\bar{s}_k = \bar{s}'_k \cdot \bar{s}''_k$$

such that

$$\alpha((a^{i'_k}u_k)(\bar{s}_ka^{j'_k})) = \alpha(a^{i'_k}, u_k, \bar{s}'_k, \alpha(\bar{s}''_k, a^{j'_k}))$$

and

$$\alpha((a^{i'_k}a_k)(\bar{s}_ka^{j'_k}) = \alpha(a^{i'_k}, a_k, \bar{s}'_k, \alpha(\bar{s}''_k, a^{j'_k})).$$

We obtain the table

$L_k^{i_{k+1}}$	$\bar{L}_k^{i_{k+1}}$
$\alpha(a^{i_k}, a_k, s_k, a^{j_k})$	$\alpha(a^{i_k}, u_k, \bar{s}'_k, \alpha(\bar{s}''_k, a^{j'_k}))$
$\alpha(a^{i'_k}, u_k, s_k, a^{j_k})$	$\alpha(a^{i_k}, a_k, \bar{s}_k, \alpha(\bar{s}''_k, a^{j'_k}))$

(12)

Now we can finish this long proof.

Let I be a Malcev sequence and $\sigma(I)$ the corresponding system of equalities in $\mathcal{S}(\mathcal{A})/\rho$ and

$$[x][y] = [u][v] \tag{13}$$

the closing equality of $\sigma(I)$.

For the system $\widetilde{\sigma}(I)$ the closing equality is

$$[a^{i_k}][x][y][a^{j_k}] = [a^{i_k}][u][v][a^{j_k}] \tag{14}$$

wich is equivalent to

$$\alpha(a^{i_k}, x, y, a^{j_k}) = \alpha(a^{i_k}, u, v, a^{j_k}) \tag{15}$$

But the last equality is the closing equality for $\widetilde{\sigma}(I)$ in terms of $\mathcal{A}$. By hypothesis, in $\mathcal{A}$ are satisfied all n-ary Malcev conditions. Consequently this equality holds. Hence, also (14) holds. $\mathcal{S}(\mathcal{A})/\rho$ being a cancellation semigroup, from (14) we get (13). Therefore $\mathcal{S}(\mathcal{A})/\rho$ is homomorphic embeddable in a group.

Malcev conditions corresponding to Malcev sequences over the subalphabet $\{L_i^1, \overline{L}_i^1, R_i^1, \overline{R}_i^1 \mid i \in \mathbb{N}\}$ of the alphabet of n-ary Malcev symbols $\{L_i^k, \overline{L}_i^k, R_i^k, \overline{R}_i^k \mid k = 1, 2, \ldots, n-1;\ i \in \mathbb{N}\}$ are called **unary Malcev conditions**.

Now we shall prove the following

Theorem 3. *If in an n-ary semigroup $\mathcal{A}$ with lateral identity are satisfied all unary Malcev conditions then $\mathcal{A}$ can be homomorphic embedded in an n-group.*

Proof. Let a_1^{n-1} be a lateral identity. For beginning we prove that $\mathcal{A}$ is cancellative.

Suppose that $\alpha(u_1^{n-1}, x) = \alpha(u_1^{n-1}, y)$. Then we have

$$\begin{aligned}\alpha(u_1^{n-1}, x) =& \alpha(u_1^{n-1}, \alpha(a_1^{n-1}, x)) = \\ =& \alpha(\alpha(u_1^{n-1}, a_1), a_2^{n-1}, x),\end{aligned}$$

and

$$x = \alpha(a_1^{n-1}, x) = \alpha(\alpha(a_1^{n-1}, a_1), a_2^{n-1}, x).$$

Then for $I = L_1^1 \overline{L}_1^1$ and

L_1^1	$\bar{L}_1^1$
$\alpha(\alpha(u_1^{n-1}, a_1), a_2^{n-1}, x)$	$\alpha(\alpha(a_1^{n-1}, a_1), a_2^{n-1}, y)$
$\alpha(\alpha(a_1^{n-1}, a_1), a_2^{n-1}, x)$	$\alpha(\alpha(u_1^{n-1}, a_1), a_2^{n-1}, y)$

(16)

we have

$$\alpha(\alpha(u_1^{n-1}, a_1), a_2^{n-1}, x) = \alpha(\alpha(u_1^{n-1}, a_1), a_2^{n-1}, y)$$

implies

$$\alpha(\alpha(a_1^{n-1}, a_1), a_2^{n-1}, y) = \alpha(\alpha(a_1^{n-1}, a_1), a_2^{n-1}, x)$$

that is

$$\alpha(u_1^{n-1}, x) = \alpha(u_1^{n-1}, y) \Rightarrow x = y.$$

Hence $\mathcal{A}$ is left cancellative.

Now from $\alpha(x, u_1^{n-1}) = \alpha(y, u_1^{n-1})$ using $I = R_1^1 \overline{R}_1^1$ and the table

R_1^1	$\bar{R}_1^1$
$\alpha(x, a_1^{n-2}, \alpha(a_{n-1}, u_1^{n-1}))$	$\alpha(y, a_1^{n-2}, \alpha(a_{n-1}, a_1^{n-1}))$
$\alpha(x, a_1^{n-2}, \alpha(a_{n-1}, a_1^{n-1}))$	$\alpha(y, a_1^{n-2}, \alpha(a_{n-1}, u_1^{n-1}))$

(17)

we get $x = y$, that is $\mathcal{A}$ is right cancellative. Consequently, $\mathcal{A}$ is a cancellative n-semigroup. We note that

L_i^k	$\bar{L}_i^k$
$\alpha(x_1^k, u_{k+1}^n)$	$\alpha(y_1^k, v_{k+1}^n)$
$\alpha(y_1^k, u_{k+1}^n)$	$\alpha(x_1^k, v_{k+1}^n)$

(18)

can be rewritten as

L_i^1	$\bar{L}_i^1$
$\alpha(\alpha(x_1^k, a_1^{n-k}), a_{n-k+1}^{n-1}, u_{k+1}^n)$	$\alpha(\alpha(y_1^k, a_1^{n-k}), a_{n-k+1}^{n-1}, v_{k+1}^n)$
$\alpha(\alpha(y_1^k, a_1^{n-k}), a_{n-k+1}^{n-1}, u_{k+1}^n)$	$\alpha(\alpha(x_1^k, a_1^{n-k}), a_{n-k+1}^{n-1}, v_{k+1}^n)$

(19)

and

R_i^k	$\bar{R}_i^k$
$\alpha(u_1^{n-k}, x_1^k)$	$\alpha(v_1^{n-k}, y_1^k)$
$\alpha(u_1^{n-k}, y_1^k)$	$\alpha(v_1^{n-k}, x_1^k)$

(20)

is equivalent to

R_i^k	$\bar{R}_i^k$
$\alpha(u_1^{n-k}, a_1^{k-1}, \alpha(a_k^{n-1}, x_1^k))$	$\alpha(v_1^{n-k}, a_1^{k-1}, \alpha(a_k^{n-1}, y_1^k))$
$\alpha(u_1^{n-k}, a_1^{k-1}, \alpha(a_k^{n-1}, y_1^k))$	$\alpha(v_1^{n-k}, a_1^{k-1}, \alpha(a_k^{n-1}, x_1^k))$

(21)

In consenquence each n-ary Malcev condition can rephrased as an unary Malcev condition.

We conclude this paper with a stand alone proof for Theorem 3.

Let be $\mathcal{A} = (A, \alpha)$ an n-semigroup and a_1^{n-1} a sequence in A such that $\alpha(x, a_1^{n-1}) = x,\ \forall x \in A$.

Zupnik proved (see [7]) that $(A, \cdot)$ where

$$x \cdot y = \alpha(x, a_1^{n-2}, y)$$

is a semigroup with a_{n-1} as a right unit,

$$x \cdot a_{n-1} = x,$$

the mapping

$$f : A \to A$$

defined by

$$xf = \alpha(a_{n-1}, x, a_1^{n-2})$$

is an endomorphism of $(A, \cdot)$ and

$$\alpha(x_1^n) = x_1 \cdot x_2 f \cdot x_3 f^2 \cdot \ldots \cdot x_n f^{n-1} \cdot a$$

where

$$a = \alpha(a_{n-1}, a_{n-1}, \ldots, a_{n-1}).$$

Suppose now that $\mathcal{A}$ is a cancellation n-semigroup. We have the following

Lemma 2. *Let be $\mathcal{A}$ a cancellative n-semigroup. The sequence a_1^{n-1} is a lateral identity iff there exists $a \in A$ such that $\alpha(a_1^{n-1}, a) = a$ or $\alpha(a, a_1^{n-1}) = a$.*

Using Lemma 2 it is easy to prove (by induction) that

Lemma 3. *In a cancellation n-semigroup any circular permutation of a lateral identity is a lateral identity too.*

Suppose now that $\mathcal{A}$ is a cancellation n-semigroup. It is easy to prove that the above endomorphism is in fact an automorphism $(yf^{-1} = \alpha(a_1^{n-2}, y, a_{n-1}))$,

$$xf^{n-1} \cdot a = a \cdot x,\ \forall x \in A$$

and

$$af = a.$$

Let now $\mathcal{A}$ be an n-semigroup with a lateral identity. We assume that all unary Malcev conditions are satisfied in $\mathcal{A}$. Then $\mathcal{A}$ is a cancellation n-semigroup (see the first part of the proof of Theorem 3). Since the table

L_k	$\bar{L}_k$	R_k	$\bar{R}_k$
$x_k s_k$	$y_k \bar{s}_k$	$w_k s_k$	$\bar{w}_k y_k$
$y_k s_k$	$x_k \bar{s}_k$	$w_k y_k$	$\bar{w}_k x_k$

(22)

in is equivalent to the table

L_k^1	$\bar{L}_k^1$	R_k^1	$\bar{R}_k^1$
$\alpha(x_k, a_2^{n-1}, s_k)$	$\alpha(y_k, a_2^{n-1}, \bar{s}_k)$	$\alpha(w_k, a_2^{n-1}, x_k)$	$\alpha(\bar{w}_k, a_2^{n-1}, y_k)$
$\alpha(y_k, a_2^{n-1}, s_k)$	$\alpha(x_k, a_2^{n-1}, \bar{s}_k)$	$\alpha(w_k, a_2^{n-1}, y_k)$	$\alpha(\bar{w}_k, a_2^{n-1}, x_k)$

(23)

in $\mathcal{A}$, it follows that the semigroup $(A, \cdot)$ is homomorphic embeddable in a group.

Let now $(\mu, (G, \cdot))$ be a free group over semigroup $(A, \cdot)$ (see [3],[4]). Then μ is a monomorphism. We extends the automorphism

$$f : (A, \cdot) \to (A, \cdot)$$

to an automorphism

$$\bar{f} : (G, \cdot) \to (G, \cdot)$$

such that

$$\mu\bar{f} = f\mu.$$

Now we have $\alpha(x_1^n)\mu\bar{f} = \alpha(x_1^n)f\mu = (x_1 \cdot x_2 f \cdot \ldots \cdot x_n f^{n-1} \cdot a)f\mu = x_1 f\mu \cdot x_2 f^2\mu \cdot \ldots \cdot x_n f^n \mu \cdot af\mu = x_1\mu\bar{f} \cdot x_2\mu\bar{f}^2 \cdot \ldots \cdot x_n\mu\bar{f}^n \cdot a\mu$.

Let be $\beta : G^n \to G$ defined by

$$\beta(y_1^n) = y_1 \cdot y_2\bar{f} \cdot \ldots \cdot y_n\bar{f}^{n-1} \cdot a\mu.$$

Then

$$\alpha(x_1^n)\mu\bar{f} = \beta(x_1\mu\bar{f}, \ldots, x_n\mu\bar{f}).$$

Finally we prove that (G, β) is an n-group. For all $x \in A$ we have

$$x\mu\bar{f}^{n-1} \cdot a\mu = xf^{n-1}\mu \cdot a\mu = (xf^{n-1} \cdot a)\mu = (a \cdot x)\mu = a\mu \cdot x\mu,$$

therefore

$$x\mu\bar{f}^{n-1} = a\mu \cdot x\mu \cdot (a\mu)^{-1}.$$

The set $A\mu$ being a generating subset of the group $(G, \cdot)$ for any $y \in G$

$$y = (x_1\mu)^{\varepsilon_1} \cdot (x_2\mu)^{\varepsilon_2} \cdot \ldots \cdot (x_k\mu)^{\varepsilon_k},$$

$x_i \in A$, $\varepsilon_i = \pm 1$ for all $i = 1, 2, \ldots, k$. Then $y\bar{f}^{n-1} = (x_1\mu\bar{f}^{n-1})^{\varepsilon_1} \cdot \ldots \cdot (x_k\mu\bar{f}^{n-1})^{\varepsilon_k} = (a\mu \cdot x_1\mu \cdot (a\mu)^{-1})^{\varepsilon_1} \cdot \ldots \cdot (a\mu \cdot x_k\mu \cdot (a\mu)^{-1})^{\varepsilon_k} = a\mu \cdot (x_1\mu)^{\varepsilon_1} \cdot (x_2\mu)^{\varepsilon_2} \cdot \ldots \cdot (x_k\mu)^{\varepsilon_k} \cdot (a\mu)^{-1} = a\mu \cdot y \cdot (a\mu)^{-1}$. From [6] it follows that (G, β) is an n-group.

In conclusion

$$\mu\bar{f} : (A, \alpha) \to (G, \beta)$$

is a homomorphic embedding.

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