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If 1+1=2 then the Pythagorean theorem holds, or one more proof of the oldest theorem of mathematics

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ABSTRACT

The Pythagorean theorem is one of the oldest theorems of mathematics. It gained during the time a central position and even today it continues to be a source of inspiration. In this note we try to give a proof which is based on a hopefully new approach. Our treatment will be as intuitive as it can be.
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One of the oldest theorems of mathematics – the Pythagorean theorem – states that the area of the square drawn on the hypotenuse (the side opposite the right angle) of a right angled triangle, is equal to the sum of the areas of the squares constructed on the other two sides, (the catheti). That is,

$$a^2 + b^2 = c^2,$$

if we denote the length of the sides of the triangle by a, b, c respectively.

Loomis ([3]) collected several hundreds of different proofs of this theorem, and classified them in categories. Maor ([4]) traces the evolution of the Pythagorean theorem and its impact on mathematics and on our culture in general. The internet page [1] containes a collection of 93 approaches to proving the theorem.

Our aim is to give a proof by a new approach, more precisely a theorem of the next type.

Theorem 1. *If the Pythagorean theorem holds in one single right angled triangle, then it holds in any right angled triangle!*

Of course we must start the proof by giving some facts we assume to be true. These will be related to the properties of a geometric transformation which we will use: the shrinking or stretching the Euclidean plane by a factor in a given direction (see figure 1).

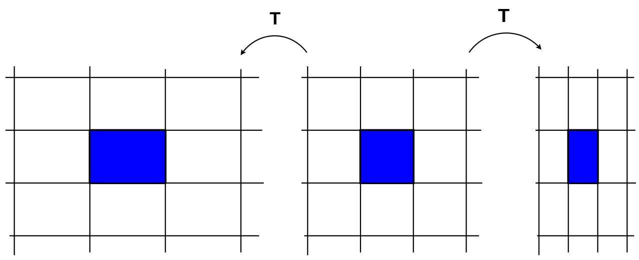

Figure 1. The transformation

Let us assume T is such a transformation having a as the shrinking/stretching factor. We can think of such a transformation as an effect of parallel projection between planes in the three-dimensional space, like in figure 2.

The transformation T obviously has the properties:

- T shrinkes/stretches the area of a plane figure by the factor a, that is if P is a plane figure, then

$$\text{Area}(T(P)) = a \cdot \text{Area}(P).$$

- It is *compatible* with the *additivity* of areas of disjoint figures, that is if P, Q are two disjoint figures then

$$\text{Area}(T(P \cup Q)) = \text{Area}(T(P)) + \text{Area}(T(Q)).$$

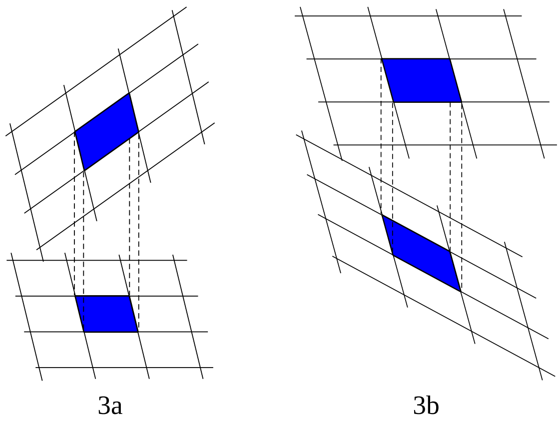

Figure 2. Streching and shrinking

- It preserves the *filling ratio* in the following sens: let P be a *regular* polygon with area A, and fix an edge L of it. Let us suppose T transforms it in a polygon $Q = T(P)$ with area aA, and suppose the image of L is K. Then the *filling ratio* defined as

$$k = \frac{\text{Area}(Q)}{\text{Area}(Q', \text{the similar polygon to } P \text{ of side } K)}$$

is independent to the length of polygon's side and the shrinking/stretching direction (see figure 3).

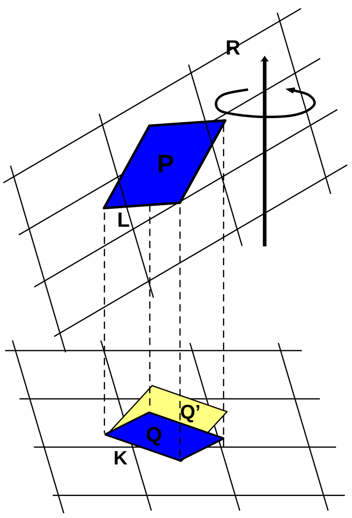

Figure 3. The invariance

Let us consider now a right angled triangle having the length of the two catheti a and b. The property asserted by the Pythagorean theorem is homogeneous in the length of the triangle edges, so we can choose the measurement unit as we like.

Now let us consider the setup of the figure 4, i.e. a regular square tiling of the plane. We will choose the measurement unit as the length of a blue square (of area 1), for a reason which will be clear soon.

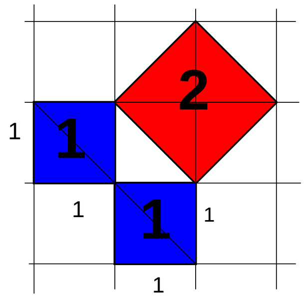

Figure 4. The special case

It is obvious that the sum of the areas of the two smaller squares is exactly the area of the third square, so *the Pythagorean theorem holds in this special case*.

Let us now apply the following transformations.

- Shrink horizontally by the factor a, see figure 5.

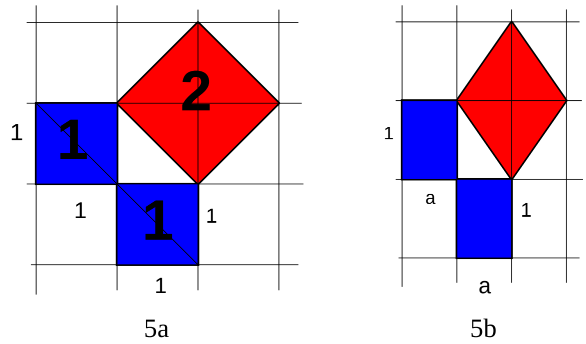

Figure 5. Shrink horizontally

- Cut at the level x, and join the piece of height $1 - x$ in an area equivalent form to the other side, according to figure 6, so we have

$$1 \cdot y = a(1 - x).$$

By this y is not a parameter, it is a function of x.

- Stretch vertically by the factor b, see figure 7.
- The sum of areas is preserved, so

$$b(a + y) + abx = 2ab$$

which is true for all x.

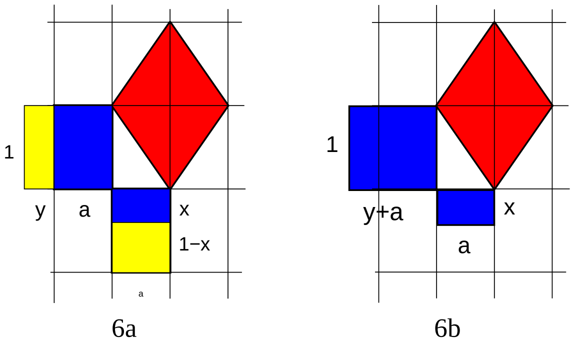

Figure 6. Cut and join

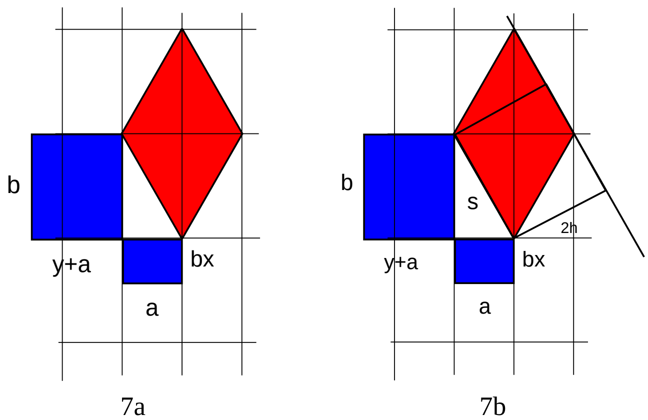

Figure 7. Stretch vertically

- By the two transformations the area of the big square changes to
$$2ab = 2sh.$$

- We can assume a, b are chosen so to have the third side of the triangle the same length s, so
$$s^2 = 2.$$

- Consequently, the filling ratio counted for the square of side s is
$$k = \frac{2h}{s} = \frac{2hs}{s^2} = hs = ab.$$

- Now the same filling ratio is valid for the two small squares, of sides a and b respectively, so now we can compute x. We first set the value of the x such that the resulting rectangles be similar. We have the condition for x:
$$\frac{a+y}{b} = \frac{bx}{a}.$$

- This equality gives
$$x = \frac{2a^2}{a^2+b^2}.$$
Now the rectangles of the base a and b are similar. The common ratio of their side length is
$$k = \frac{bx}{a} = \frac{a+y}{b} = \frac{2ab}{a^2+b^2},$$
and this is the common filling ratio (smaller than 1) by which the sum of the area of the two smaller squares is changed and also the area of the third square, too.

- By the shrinking factor a and stretching factor b all the area are changed by the factor ab as we have seen already, so we have
$$\frac{2ab}{a^2+b^2} = ab$$
and consequently
$$a^2 + b^2 = 2.$$

This last equation simply means that *the Pythagorean theorem is valid in any right angled triangle.* QED.

References

[1] http://www.cut-the-knot.org/pythagoras/index.shtml

[2] W. J. Hazard, *Generalizations Of The Theorem Of Pythagoras And Euclid'S Theorem Of The Gnomon*, Am Math Monthly, vol. 36, n. 1, 32-34, (1929)

[3] E. S. Loomis, *The Pythagorean Proposition*, NCTM, 1968.

[4] E. Maor, *The Pythagorean Theorem – A 4,000-Year History*, Princeton University Press, 2007.

[5] R. B. Nelsen, *Proofs Without Words*, MAA, 1993.

[6] R. B. Nelsen, *Proofs Without Words II*, MAA, 2000.

[7] B. F. Yanney and J. A. Calderhead, *New and Old Proofs of the Pythagorean Theorem*, Am Math Monthly,
vol. 3, (1896),
n. 4, 110-113, n. 6/7, 169-171, n. 12, 299-300,
vol. 4, (1897),
n. 1, 11-12, n. 3, 79-81, n. 6/7, 168-170, 250-251, 267-269,
vol. 6, (1899),
n. 2, 33-34.