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EXISTENCE OF POSITIVE SOLUTIONS FOR A FOURTH–ORDER DIFFERENTIAL SYSTEM WITH VARIABLE COEFFICIENTS

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ABSTRACT

This paper investigates the existence of positive solutions for a fourth-order differential system using a fixed point theorem of cone expansion and compression type. The two main results give sufficient conditions to insure at least one and at least two positive solutions, respectively.

Keywords: positive solutions, Green's function, boundary value problems, fixed point theorem, complete continuity

1 Introduction

It is well known that the bending of an elastic beam can be described with fourth-order boundary value problems. An elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t), u''(t)\big), & 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1}$$

The existence of solutions for problem (1) was established for example by Aftabizadeh [1], Gupta [4, 5], Liu [6], Ma [7], Ma et. al. [8], Ma and Wang [9], Del Pino and Manasevich [10], Yang [11] (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions.

Recently, Wang and An [12] studied the existence of positive solutions for the second-order boundary value problem

$$\begin{cases} -u'' + \lambda u = u\varphi + f(t, u), & 0 < t < 1 \\ -\varphi'' = \mu u, & 0 < t < 1 \\ u(0) = u(1) = 0 \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{2}$$

where $\lambda > -\pi^2$, μ is a positive parameter, and $f(t,u) : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

In this paper we discuss the existence of positive solutions for the fourth-order boundary value problem

$$\begin{cases} u^{(4)} + A(t)u'' - B(t)u = \varphi u + f(t, u, u''), & 0 < t < 1 \\ -\varphi'' = \mu u, 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0 \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{3}$$

where $A, B \in C[0,1]$ and μ is a positive parameter. The existence of the positive solution depends on μ, i.e. there exists a positive number $\bar{\mu}$, such that if $0 < \mu < \bar{\mu}$, the BVP (3) has a positive solution. For this, we shall assume the following conditions throughout:

(A1) $f(t, u, v) : [0,1] \times [0,\infty) \times (-\infty, 0] \longrightarrow [0,\infty)$ is continuous;

(A2) $b = \inf_{t\in[0,1]} A(t) > -\pi^2$, $c = \inf_{t\in[0,1]} B(t) > 0$, $\pi^4 - b\pi^2 - c > 0$, where $b, c \in \mathbb{R}$, $b = -\lambda_1 -$

$\lambda_2 < 2\pi^2$, $c = -\lambda_1\lambda_2 \geq 0$, and $\lambda_1 \geq 0 \geq \lambda_2 > -\pi^2$.

Assumption (A2) involves a two-parameter nonresonance condition.

In fact as we will see below one could consider in Section 2 and 3 that $f(t,u,v) = g(t) \cdot h(t,u,v)$ with $h(t,u,v) : [0,1] \times [0,\infty) \times (-\infty, 0] \longrightarrow [0,\infty)$ continuous and $g \in C([0,1], \mathbb{R}_+)$ provided

$$\int_0^1 \int_0^1 G_1(\tau,\tau)\, G_2(\tau,s)\, g(s)\, ds\, d\tau < +\infty;$$

here G_1, G_2 are as defined in Section 2.

2 Preliminaries

Let $Y = C[0,1]$ and $Y_+ = \{u \in Y \,|\, u(t) \geq 0, t \in [0,1]\}$. It is well known that Y is a Banach space equipped with the usual Cebîşev norm $\|u\|_0 = \sup_{t\in[0,1]} |u(t)|$. Let us denote by $\|\cdot\|_2$ the norm

$$\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}.$$

It is easy to show that $C^2[0,1]$ is complete with the norm $\|\cdot\|_2$ and $\|u\|_2 \leq \|u\|_0 + \|u''\|_0 \leq 2\|u\|_2$.

Let us set $X = C_0^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$. For given $\lambda \geq 0$, denote the norm $\|\cdot\|_\lambda$ by $\|u\|_\lambda = \sup_{t\in[0,1]}\{|u''(t)| + \lambda|u(t)|\}$, $u \in X$. It can be shown that $(X, \|\cdot\|_\lambda)$ and $(X, \|\cdot\|_2)$ are both Banach spaces ([2]).

We need the following ten lemmas.

Lemma 1. ([2]) $\forall u \in X$, $\|u\|_0 \leq \|u''\|_0$.

Lemma 2. ([2]) $\forall u \in X$ *one has*

$$(1+\lambda)^{-1}\|u\|_\lambda \leq \|u\|_2 \leq \|u\|_\lambda.$$

Suppose that $G_i(t,s)$, $i \in \{1,2,3\}$ is the Green function associated to

$$-u'' + \lambda_i u = 0, \;\; u(0) = u(1) = 0.$$

Let $\omega_i = \sqrt{|\lambda_i|}$, then $G_i(t,s)$, $i \in \{1,2,3\}$ can be expressed as

(i) when $\lambda_i > 0$, $G_i(t,s) =$

$$\begin{cases} \dfrac{\sinh \omega_i t \cdot \sinh \omega_i(1-s)}{\omega_i \sinh \omega_i}, & 0 \leq t \leq s \leq 1 \\ \dfrac{\sinh \omega_i s \cdot \sinh \omega_i(1-t)}{\omega_i \sinh \omega_i}, & 0 \leq s \leq t \leq 1; \end{cases}$$

(ii) when $\lambda_i = 0$,

$$G_i(t,s) = K(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1; \end{cases}$$

(iii) when $-\pi^2 < \lambda_i < 0$, $G_i(t,s) =$

$$\begin{cases} \dfrac{\sin \omega_i t \cdot \sin \omega_i(1-s)}{\omega_i \sin \omega_i}, & 0 \leq t \leq s \leq 1 \\ \dfrac{\sin \omega_i s \cdot \sin \omega_i(1-t)}{\omega_i \sin \omega_i}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Lemma 3. $G_i(t,s)$, $i \in \{1,2,3\}$ *has the following properties:*

(i) $G_i(t,s) > 0, \forall t,s \in (0,1)$;

(ii) $G_i(t,s) \leq C_i \cdot G_i(s,s), \forall t,s \in [0,1]$;

(iii) $G_i(t,s) \geq \delta_i G_i(t,t) G_i(s,s), \forall t,s \in [0,1]$,

where $C_i = 1, \delta_i = \frac{\omega_i}{\sinh \omega_i}$, *if* $\lambda_i > 0$; $C_i = 1$, $\delta_i = 1$, *if* $\lambda_i = 0$; $C_i = \frac{1}{\sin \omega_i}$, $\delta_i = \omega_i \sin \omega_i$, *if* $-\pi^2 < \lambda_i < 0$.

For $h \in Y$, let us consider the following linear boundary value problem:

$$-u'' + \lambda_i u = h(t), \;\; u(0) = u(1) = 0. \quad (4)$$

Then the solution of (4) can be expressed as

$$u(t) = \int_0^1 G_i(t,s)\, h(s)\, ds, \;\; i \in \{1,2\}. \quad (5)$$

We now define a mapping $T_i : C[0,1] \to C[0,1]$ by

$$(T_i h)(t) = \int_0^1 G_i(t,s) h(s)\, ds, \;\; i \in \{1,2\}. \quad (6)$$

Using Lemma 3 we have

$$|(T_i h)(t)| = \left| \int_0^1 G_i(t,s) h(s)\, ds \right|$$

$$\leq C_i \|h\|_0 \int_0^1 G_i(s,s)\, ds \leq C_i D_i \|h\|_0 = M_i \|h\|_0,$$

where $M_i = C_i D_i$, $D_i = \displaystyle\int_0^1 G_i(s,s)\, ds$. Thus $\|T_i h\|_0 \leq M_i \|h\|_0$, and therefore

$$\|T_i\| \leq M_i, i \in \{1,2\}. \quad (7)$$

Lemma 4. ([12]) *Let E be a real Banach space and let $P \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $Q : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Qu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Qu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; *or*

(ii) $\|Qu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Qu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then, Q has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 5. *Let $f_n : (0,1) \to \mathbb{R}$ be a sequence of a continuously differentiable functions. If*

i) $\lim_{n\to\infty} f_n(x) = f(x)$ *on* $(0,1)$;

ii) $\lim_{n\to\infty} f'_n(x) = p(x)$, *where the convergence is uniformly on* $(0,1)$,

then, f is continuously differentiable on $(0,1)$, and for all $x \in (0,1)$ one has

$$\lim_{n\to\infty} f'_n(x) = f'(x).$$

For $h \in Y$, consider the following linear boundary value problem:

$$\begin{cases} u^{(4)} + bu'' - cu = h(t), \quad 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{8}$$

where b, c satisfy the assumption

$$\pi^4 - b\pi^2 - c > 0 \tag{9}$$

and let $\Gamma = \pi^4 - b\pi^2 - c$. The inequality (9) follows immediately from the fact that $\Gamma = \pi^4 - b\pi^2 - c$ is the first eigenvalue of the problem $u^{(4)} + bu'' - cu = \lambda u$, $u(0) = u(1) = u''(0) = u''(1) = 0$ and $\phi_1(t) = \sin \pi t$ is the first eigenfunction, i.e. $\Gamma > 0$.

Let us consider the polynomial $P(\lambda) = \lambda^2 + b\lambda - c$, where $b < 2\pi^2, c \geq 0$. It is easy to see that P has two real roots $\lambda_1, \lambda_2 = \frac{-b\pm\sqrt{b^2+4c}}{2}$, with $\lambda_1 \geq 0 \geq \lambda_2 > -\pi^2$. In this case, (8) satisfies the following decomposition form:

$$\begin{aligned} & u^{(4)} + bu'' - cu \\ = \;& \Big(-\frac{d^2}{dt^2} + \lambda_1\Big)\Big(-\frac{d^2}{dt^2} + \lambda_2\Big)u, \; 0 < t < 1. \end{aligned} \tag{10}$$

It is obvious that $b = -\lambda_1 - \lambda_2 < 2\pi^2$, $c = -\lambda_1\lambda_2 \geq 0$.

Now, since

$$\begin{aligned} & u^{(4)} + bu'' - cu \\ = \;& \Big(-\frac{d^2}{dt^2} + \lambda_1\Big)\Big(-\frac{d^2}{dt^2} + \lambda_2\Big)u \\ = \;& \Big(-\frac{d^2}{dt^2} + \lambda_2\Big)\Big(-\frac{d^2}{dt^2} + \lambda_1\Big)u = h(t), \end{aligned}$$

the solution of the boundary value problem (8) can be expressed by

$$u(t) = \int_0^1 \int_0^1 G_1(t,v)G_2(v,s)h(s)\,dsdv, \quad t \in [0,1]. \tag{11}$$

Thus, for every given $h \in Y$, the boundary value problem (8) has a unique solution $u \in C^4(0,1)$ given by (11).

We now define a mapping $T : C[0,1] \to C[0,1]$ by

$$\begin{aligned} &(Th)(t) = (T_2T_1h)(t) = \\ &\int_0^1 \int_0^1 G_1(t,v)G_2(v,s)h(s)\,ds\,dv, \quad t \in [0,1]. \end{aligned} \tag{12}$$

From (5) and (6) one can obtain the following result.

Lemma 6. ([2]) *The operator*

$$T : C[0,1] \to (X, \|\cdot\|_{\lambda_1})$$

is linear completely continuous, and $\|T\| \leq D_2$.

Throughout this article we denote by $Th = u$ the unique solution of the linear boundary value problem (8).

The boundary value problem

$$-\varphi'' = \mu u, \quad \varphi(0) = \varphi(1) = 0,$$

can be solved by using the Green's function, namely,

$$\varphi(t) = \mu \int_0^1 K(t,s)u(s)\,ds, \quad 0 < t < 1. \tag{13}$$

Thus, inserting (13) into the first equation of (3), we have

$$\begin{cases} u^{(4)} + B(t)u'' - A(t)u \\ = \mu u(t) \displaystyle\int_0^1 K(t,s)u(s)\,ds + f(t,u,u'') \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{14}$$

Now we consider the existence of a positive solution of (14). The function $u \in C^4(0,1) \cap C^2[0,1]$ is a positive solution of (14), if $u \geq 0$, $t \in [0,1]$, and $u \not\equiv 0$.

Let us consider the following linear boundary value problem:

$$\begin{cases} u^{(4)} + bu'' - cu \\ = \mu u(t) \displaystyle\int_0^1 K(t,s)u(s)\,ds + f(t,u,u'') \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{15}$$

Then, the solution of (15) can be expressed as

$$\begin{aligned} & u(t) = \mu \int_0^1 \int_0^1 G_2(t,\tau)\, G_1(\tau,s)\, u(s) \\ & \int_0^1 K(s,v)\, u(v)\, dv\, ds\, d\tau \\ & + \int_0^1 \int_0^1 G_2(t,\tau)\, G_1(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau \end{aligned}$$

and its second-order derivative can be expressed by

$$u''(t) = \lambda_2 u(t) - \mu \int_0^1 G_1(t,s)\, u(s) \int_0^1 K(s,v)\, u(v)\, dv\, ds - \int_0^1 G_1(t,s)\, f(s, u(s), u''(s))\, ds.$$

Let us set the cone in $C^2[0,1]$

$$P_2 = \{u \in C^2[0,1] : u(0) = u(1) = 0, \; u \geq 0,\; u'' \leq 0 \text{ on } [0,1]\}$$

and

$$P = P_2 \cap \{u \in C^2[0,1] :\; u(t) \geq \sigma_1 \|u\|_0, \; -u''(t) \geq \sigma_2 \|u''\|_0, t \in \left[\frac{1}{4}, \frac{3}{4}\right] \},$$

where

$$\sigma_1 = \frac{\delta_1}{C_1}(1-L) \min_{t \in [\frac{1}{4},\frac{3}{4}]} G_1(t,t),$$

$$\sigma_2 = \frac{\delta_1}{C_m C_1} \min_{t \in [\frac{1}{4},\frac{3}{4}]} G_1(t,t), \text{ and}$$

$$C_m = \max\left\{\frac{1}{1-L}, \frac{1}{1-L_1}\right\}.$$

It is easy to check that P is a cone in $C^2[0,1]$ as well.

For $R > 0$, write

$$B_R = \{u \in C^2[0,1] : \|u\|_2 < R\}.$$

Consider the following boundary value problem (see [2]):

$$\begin{cases} u^{(4)} + bu'' - cu = -(A(t) - b)\, u'' + (B(t) - c)\, u + h(t) \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \quad (16)$$

For any $u \in X$, let

$$Gu = -(A(t) - b)\, u'' + (B(t) - c)\, u.$$

The operator $G : X \to Y$ is linear. By Lemmas 2 and 3, $\forall u \in X, t \in [0,1]$, we have

$$|(Gu)(t)| \leq [B(t) + A(t) - (b+c)]\, \|u\|_2 \leq K\|u\|_2 \leq K\|u\|_{\lambda_2},$$

where $K = \max_{t \in [0,1]} [B(t) + A(t) - (b+c)]$. Hence $\|Gu\|_0 \leq K\|u\|_{\lambda_2}$, and so $\|G\| \leq K$. Also $u \in C^2[0,1] \cap C^4(0,1)$ is a solution of (16) iff $u \in X$ satisfies $u = T(Gu + h)$, i.e.

$$u \in X, \quad (I - TG)\, u = Th. \quad (17)$$

The operator $I - TG$ maps X into X. From $\|T\| \leq D_2$ together with $\|G\| \leq K$ and condition $L := D_2 K < 1$, and applying the operator spectral theorem, we find that $(I - TG)^{-1}$ exists and it is bounded.

Let $H = (I - TG)^{-1} T$. Then (17) is equivalent to $u = Hh$. By the Neumann expansion formula, H can be expressed by

$$\begin{aligned} H &= (I + TG + \ldots + (TG)^n + \ldots)\, T \\ &= T + (TG)T + \ldots + (TG)^n T + \ldots \end{aligned} \quad (18)$$

The complete continuity of T with the continuity of $(I - TG)^{-1}$ guarantees that the operator $H : Y \to X$ is completely continuous.

Now $\forall h \in Y_+$, let $u = Th$, then $u \in X \cap Y_+$, and $u'' \leq 0$. Thus we have

$$(Gu)(t) = -(B(t) - c)\, u'' + (A(t) - b)\, u \geq 0, \quad t \in [0,1].$$

Hence

$$\forall\, h \in Y_+, \quad (GTh)(t) \geq 0, \quad t \in [0,1] \quad (19)$$

and so $(TG)\,(Th)\,(t) = T\,(GTh)\,(t) \geq 0$, $t \in [0,1]$.

It is easy to see ([2]) that the following inequalities hold, $\forall h \in Y_+$

$$\frac{1}{1-L}(Th)(t) \geq (Hh)(t) \geq (Th)(t), \; t \in [0,1],$$

and

$$\|Hh\|_0 \leq \frac{1}{1-L}\|Th\|_0.$$

If we now introduce the following notation

$$V_1(t) = \Big(-\frac{d^2}{dt^2} + \lambda_2\Big) u,$$

then, from (16), using (10), we have

$$\Big(-\frac{d^2}{dt^2} + \lambda_1\Big) V_1 = Gu + h(t) \quad (20)$$

$$V_1(0) = 0, \;\; V_1(1) = 0.$$

So, the following boundary value problem

$$\begin{cases} -u''(t) + \lambda_2 u(t) = V_1(t), \\ u(0) = u(1) = 0 \end{cases}$$

can be solved by

$$u(t) = (T_2 V_1)(t) = \int_0^1 G_2(\tau, s) V_1(s)\, ds. \quad (21)$$

Moreover, from (20) using (21) we obtain

$$\Big(-\frac{d^2}{dt^2} + \lambda_1\Big) V_1 = G T_2 V_1 + h(t) \quad (22)$$

$$V_1(0) = 0, \qquad V_1(1) = 0. \tag{23}$$

From equation (22), we have

$$V_1(t) = T_1\big(GT_2V_1 + h(t)\big).$$

On the other hand, $V_1 \in C[0,1] \cap C^2(0,1)$ is a solution of (22-23) iff $V_1(t)$ satisfies $V_1 = T_1\left(GT_2V_1 + h\right)$, i.e.

$$(I - T_1GT_2)V_1 = T_1h. \tag{24}$$

From $\|T_1\| \le M_1$, $\|T_2\| \le M_2$ together with $\|G\| \le K$ and condition $M_1M_2K < 1$, applying the operator spectra theorem, we have that $(I - T_1GT_2)^{-1}$ exists and it is bounded. Let $L_1 = M_1M_2K$.

Let $H_1 = (I - T_1GT_2)^{-1}T_1$. Then, (24) is equivalent to $V_1 = H_1h$. By the Neumann expansion formula, H_1 can be expressed by

$$\begin{aligned} H_1 &= (I + T_1GT_2 + (T_1GT_2)^2 \\ &\quad + \ldots + (T_1GT_2)^n + \ldots)T_1 \\ &= T_1 + (T_1GT_2)T_1 + (T_1GT_2)^2T_1 \\ &\quad + \ldots + (T_1GT_2)^nT_1 + \ldots \end{aligned} \tag{25}$$

The complete continuity of T_1 with the continuity of $(I - T_1GT_2)^{-1}$ yields that the operator $H_1 : Y \to C^2[0,1]$ is completely continuous.

By (25) we have that $\forall\, h \in Y_+$,

$$\begin{aligned} (H_1h)(t) &= (T_1h)(t) + \big((T_1GT_2)T_1h\big)(t) \\ &\quad + \big((T_1GT_2)^2T_1h\big)(t) \\ &\quad + \ldots + \big((T_1GT_2)^nT_1h\big)(t) + \ldots \\ &\ge (T_1h)(t), \quad t \in [0,1], \end{aligned}$$

and so $H_1 : Y_+ \to Y_+ \cap C^2[0,1]$.

On the other hand, we have that $\forall\, h \in Y_+$,

$$\begin{aligned} (H_1h)(t) &\le (Th)(t) + \|T_1GT_2\|(T_1h)(t) \\ &\quad + \|T_1GT_2)\|^2(T_1h)(t) + \ldots + \\ &\quad \|T_1GT_2\|^n(T_1h)(t) + \ldots \\ &\le (1 + L_1 + \ldots + L_1^n + \ldots) \\ &\quad (T_1h)(t) = \frac{1}{1 - L_1}(T_1h)(t). \end{aligned}$$

So, the following inequalities hold:

$$(H_1h)(t) \le \frac{1}{1 - L_1}\|T_1h\|_0, \ t \in [0,1]$$

and

$$\|H_1h\|_0 \le \frac{1}{1 - L_1}\|T_1h\|_0.$$

Hence

$$\frac{1}{1 - L_1}(T_1h)(t) \ge (H_1h)(t) \ge (T_1h)(t), \ t \in [0,1].$$

For any $u \in Y_+ \cap C^2[0,1]$, let

$$Fu(t) = \mu u(t)\int_0^1 K(t,s)u(s)\,ds + f(t,u,u''). \tag{26}$$

From (A1), we have that $F : Y_+ \cap C^2[0,1] \to Y_+$ is continuous. It is easy to see that $u \in C^2[0,1] \cap C^4(0,1)$ being a positive solution of (14) is equivalent to $u \in Y_+$ being a nonzero solution of

$$u = HFu. \tag{27}$$

Let $Q = HF$. Obviously, $Q : Y_+ \cap C^2[0,1] \to Y_+ \cap C^2[0,1]$ is completely continuous. We next show that the operator Q has a nonzero fixed point in $Y_+ \cap C^2[0,1]$.

From (12) and (26) we also have

$$\begin{aligned} TFu(t) &= T_2T_1\left(\mu u(s)\int_0^1 K(s,v)\,u(v)\,dv\right. \\ &\qquad f(s,u(s),u''(s))) \\ &= \mu\int_0^1\int_0^1 G_2(t,\tau)\,G_1(\tau,s)\,u(s) \\ &\qquad \int_0^1 K(s,v)\,u(v)\,dv\,ds\,d\tau \\ &\quad + \int_0^1\int_0^1 G_2(t,\tau)\,G_1(\tau,s)\,f(s,u(s),u''(s))\,ds\,d\tau \end{aligned} \tag{28}$$

and hence

$$\begin{aligned} (TFu)''(t) &= \lambda_2TFu(t) - \mu\int_0^1 G_1(t,s)\,u(s) \\ &\qquad \int_0^1 K(s,v)\,u(v)\,dv\,ds\,d\tau - \\ &\qquad \int_0^1 G_1(t,s)\,f(s,u(s),u''(s))\,ds\,d\tau. \end{aligned}$$

So, from (18) and (28), we have

$$\begin{aligned} (Qu)(t) = (HFu)(t) &= (TFu)(t) + \big((TG)TFu\big)(t) \\ &\quad + \ldots + \big((TG)^nTFu\big)(t) + \ldots \end{aligned} \tag{29}$$

Lemma 7. *Let $u \in P$. Then, the following relations hold:*

(a) $(Qu)(t) \ge \dfrac{\delta_1}{C_1}(1 - L) \cdot G_1(t,t)\|Qu\|_0$, *for* $t \in [0,1]$;

(b) $-(Qu)''(t) \ge \dfrac{\delta_1}{C_mC_1} \cdot G_1(t,t)\|(Qu)''\|_0$, *for* $t \in [0,1]$.

Proof. From Lemma 3 it is easy to see that

$$\begin{aligned} Qu(t) &\le \frac{1}{1 - L}(TFu)(t) \le \frac{C_1}{1 - L}\int_0^1 \\ &\qquad \int_0^1 G_1(\tau,\tau)G_2(\tau,s)Fu(s)\,ds d\tau, \ t \in [0,1] \end{aligned}$$

and so

$$\|Qu\|_0 \le \frac{C_1}{1-L} \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) Fu(s)\, ds d\tau.$$

Hence

$$\begin{aligned} Qu(t) &\ge (TFu)(t) \ge \delta_1 G_1(t,t) \int_0^1 \int_0^1 \\ &\quad G_1(\tau,\tau) G_2(\tau,s) Fu(s)\, ds d\tau n \\ &\ge \frac{\delta_1(1-L) G_1(t,t)}{C_1} \ge \|Qu\|_0. \end{aligned} \tag{30}$$

Similarly, it is easy to see that

$$-(Qu)''(t) \ge \frac{\delta_1}{\widehat{C} C_1} G_1(t,t) \|(Qu)''\|_0. \tag{31}$$

Indeed, using (18) H can be expressed by

$$\begin{aligned} Hh &= (I + TG + (TG)^2 \\ &\quad + \ldots + (TG)^n + \ldots)\, Th \\ &= Th + TGTh + (TG)^2 Th \\ &\quad + \ldots + (TG)^n Th + \ldots \\ &= T(Ih + GTh + (GT)^2 h \\ &\quad + \ldots + (GT)^n h + \ldots). \end{aligned} \tag{32}$$

If we differentiate the right hand side of (18) with the help of (32), $\forall h \in Y_+$ we have the following:

$$\begin{aligned} &T'(Ih + GTh + (GT)^2 h \\ &\quad + \ldots + (GT)^n h + \ldots) \\ &= T'h + T'G(Th + (TG)Th \\ &\quad + \ldots + (TG)^n Th + \ldots) \\ &\le T'h + T'G(Th + \|TG\| Th \\ &\quad + \ldots + \|TG\|^n Th + \ldots) \\ &\le T'h + T'G(1 + L + \ldots + L^n + \ldots) Th \\ &= T'h + \frac{1}{1-L} (T'G) Th. \end{aligned}$$

Then the series

$$T'h + T'GTh + T'G(TG)Th + \ldots + T'G(TG)^n Th + \ldots$$

uniformly converges on $(0,1)$.

Using Lemma 5, if we differentiate both sides of (18), we get

$$\begin{aligned} (Hh)' &= T'h + T'GTh + T'G(TG)Th \\ &\quad + \ldots + T'G(TG)^n Th + \ldots \end{aligned} \tag{33}$$

Similarly, using Lemma 5 it is also true that

$$\begin{aligned} (Hh)'' &= T''h + T''GTh + T''G(TG)Th \\ &\quad + \ldots + T''G(TG)^n Th + \ldots, \end{aligned} \tag{34}$$

because the series

$$T''h + T''GTh + T''G(TG)Th + \ldots + T''G(TG)^n Th + \ldots$$

also uniformly converges on $(0,1)$. If we differentiate both sides of (33), we find equation (34).

Finally, we differentiate twice both sides of equation (12) with respect to t in order to find T''

$$\begin{aligned} (Th)''(t) &= \lambda_2 (Th)(t) - \int_0^1 G_1(t,s) h(s)\, ds \\ &= \lambda_2 (Th)(t) - (T_1 h)(t), \ t \in [0,1]. \end{aligned} \tag{35}$$

Using (34) and (35) we obtain

$$(Hh)''(t) = \lambda_2 (Hh)(t) - (H_1 h)(t)$$

with Hh and $H_1 h$ from (18) and (25), respectively. Let $h = F(u)$, then we obtain

$$(Qu)''(t) = (HF(u))''(t) = \lambda_2 (HF(u))(t) - (H_1 F(u))(t).$$

The proof of (31) is similar to the proof of (30).

Similarly,

$$\begin{aligned} -(Qu)''(t) &= (-\lambda_2)(HFu)(t) + (H_1 Fu)(t) \\ &\le (-\lambda_2) \frac{1}{1-L} (TFu)(t) + \frac{1}{1-L_1} (T_1 Fu)(t) \\ &= \frac{(-\lambda_2)}{1-L} \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) Fu(s)\, ds d\tau \\ &\quad + \frac{1}{1-L_1} \int_0^1 G_1(t,s) Fu(s)\, ds \\ &\le C_1 C_m \left(-\lambda_2 \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) \right. \\ &\quad \left. Fu(s)\, ds d\tau + \int_0^1 G_1(s,s) Fu(s)\, ds \right), \ t \in [0,1] \end{aligned}$$

and so

$$\begin{aligned} \|(Qu)''\|_0 &\le C_1 C_m \left(-\lambda_2 \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) \right. \\ &\quad \left. Fu(s)\, ds d\tau + \int_0^1 G_1(s,s) Fu(s)\, ds \right). \end{aligned}$$

Hence

$$\begin{aligned} -(Qu)''(t) &\ge (-\lambda_2)(TFu)(t) + (T_1 Fu)(t) \\ &\ge \delta_1 G_1(t,t) \left((-\lambda_2) \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) \right. \\ &\quad \left. Fu(s)\, ds d\tau + \int_0^1 G_1(s,s) Fu(s)\, ds \right) \\ &\ge \frac{\delta_1 G_1(t,t)}{C_1 C_m} \|(Qu)''\|_0. \end{aligned}$$

∎

Throughout this paper, we assume additionally that the function $f(t,u,v)$ satisfies

$$\begin{aligned} (H1) \quad & f(t,u,v) \le f_1(t) f_2(|u| + |v|), \\ & t \in (0,1), \ u \in \mathbb{R}_+, v \in \mathbb{R}_-, \end{aligned}$$

where $f_1 \in C([0,1], \mathbb{R}_+)$, $f_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$.

Let us introduce the following notations

$$D_1 = \int_0^1 \int_0^1 G_1(\tau,\tau)\, K(\tau,s)\, ds\, d\tau,$$
$$D_2 = \int_0^1 G_1(s,s)\, f_1(s)\, ds,$$
$$D_3 = \int_0^1 \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) K(s,v)\, dv ds d\tau,$$
$$D_4 = \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) f_1(s)\, ds d\tau,$$
$$D_5 = \int_{\frac14}^{\frac34} \int_{\frac14}^{\frac34} G_2(\frac12,\tau) G_1(\tau,s)\, ds d\tau.$$

Lemma 8. *Suppose that $(H1)$ applies. Then, for all $u \in C^2[0,1]$ such that $u(0) = u(1) = 0$, $u \geq 0$, and $u'' \leq 0$, the following hold*

$$(TFu)(t) \leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|), \quad t \in (0,1),$$

and

$$-(TFu)''(t) \leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|), \; t \in (0,1).$$

Proof. It is easy to see that $D_3 \leq D_1$ and $D_4 \leq D_2$.

By Lemma 1 and $(H1)$ we have

$$\begin{aligned} TFu(t) \leq \mu &\int_0^1 \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) K(s,v)\; dv\, ds\, d\tau \|u\|_0^2 \\ &+ \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s)\, f_1(s)\, ds\, d\tau \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|) \\ &\leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2\left(|u(s)| + |u''(s)|\right), \end{aligned}$$

and similarly we also have

$$\begin{aligned} &-(TFu)''(t) \leq \mu \int_0^1 \int_0^1 K(\tau,\tau)\, K(\tau,s)\, ds\, d\tau \|u\|_0^2 \\ &+ \int_0^1 K(s,s)\, f_1(s)\, ds\, d\tau \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|) \\ &\leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|). \end{aligned}$$

∎

Lemma 9. *$Q(P) \subset P$ and $Q : P \to P$ is completely continuous.*

Proof. Let $u \in P$, then we define the mapping $Q : P \to C^2[0,1]$ by (29). Then, for any $u \in P$, it is clear that

$$\begin{aligned} (TFu)''(t) = \lambda_2 TFu(t) - \mu &\int_0^1 G_1(t,s)\, u(s) \int_0^1 K(s,v)\, u(v)\, dv\, ds\, d\tau \\ &- \int_0^1 G_1(t,s)\, f(s,u(s),u''(s))\, ds\, d\tau \leq 0, \end{aligned} \tag{36}$$

because $\lambda_2 \leq 0$. So, using (34) and (36), we have

$$\begin{aligned} (Qu)''(t) &= (HF(u))''(t) \\ &= \lambda_2 (HF(u))(t) - (H_1F(u))(t) \leq 0. \end{aligned}$$

By Lemma 7,

$$\begin{aligned} (Qu)(t) &\geq \frac{\delta_1}{C_1}(1-L) G_1(t,t) \|Qu\|_0 \\ &\geq \sigma_1 \|Qu\|_0, \quad t \in \left[\frac14, \frac34\right] \end{aligned}$$

and

$$\begin{aligned} -(Qu)''(t) &\geq \frac{\delta_1}{C_m C_1} G_1(t,t) \|Qu''\|_0 \\ &\geq \sigma_2 \|Qu\|_0, \; t \in \left[\frac14, \frac34\right]. \end{aligned}$$

Hence $Q(P) \subset P$.

Let $V \subset P$ be a bounded set. Then there exists $d > 0$, such that $\sup\{\|u\|_2 : u \in V\} = d$.

First we prove $Q(V)$ is bounded. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$, we have $|u(t)| + |u''(t)| \leq \|u\|_0 + \|u''\|_0 \leq 2d$, for all $t \in [0,1]$. Let $M_d = \sup\{f_2(w) : w \in [0,2d]\}$. Now, from Lemma 3 we have for any $u \in V$ and $t \in [0,1]$ that

$$\begin{aligned} |TFu(t)| = \Big|\mu &\int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, u(s) \int_0^1 K(s,v)\, u(v)\, dv\, ds\, d\tau + \int_0^1 \int_0^1 K(t,\tau) \\ &K(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau\ \Big| \\ &\leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|) \leq \mu D_1 d^2 + M_d D_2. \end{aligned} \tag{37}$$

Using (37), we obtain $(TFu)\|_0 \leq \mu D_1 d^2 + M_d D_2$ and

$$\|HFu\|_0 \leq \frac{1}{1-L} \|TFu\|_0 \leq \frac{1}{1-L}(\mu D_1 d^2 + M_d D_2).$$

We have a similar type of inequality for $|(Tu)''(t)|$ and $\|\,(HFu)''\,\|_0$.

Therefore $Q(V)$ is bounded.

Next we prove that $Q(V)$ is equicontinuous. Now from Lemma 4 we have for any $u \in V$ and any $t_1, t_2 \in [0,1]$ that

$$\begin{aligned}
&|(TFu)(t_1) - (TFu)(t_2)| \le \mu \int_0^1 \int_0^1 |K(t_1,\tau) \\
&- K(t_2,\tau)|\, K(\tau,s)\, u(s) \int_0^1 K(s,v)\, u(v)\, dv\, ds\, d\tau \\
&+ \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)| K(\tau,s) \\
&\quad f(s, u(s), u''(s))\, ds\, d\tau \\
&\le \mu \int_0^1 \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)| \\
&K(\tau,s)\, K(s,v)\, dv\, ds\, d\tau \|u\|_0^2 \\
&+ \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)| K(\tau,s)\, f_1(s)\, f_2(|u(s)| \\
&+ |u''(s)|)\, ds\, d\tau \\
&\le \mu |t_1 - t_2| \int_0^1 \int_0^1 K(s,s)\, K(s,v)\, dv\, ds\, \|u\|_0^2 \\
&+ M_d |t_1 - t_2| \int_0^1 K(s,s)\, f_1(s)\, ds \\
&\le (\mu D_1 d^2 + M_d D_3)|t_1 - t_2|.
\end{aligned} \tag{38}$$

Using (18), we have

$$\begin{aligned}
&|(Hh)(t_1) - (Hh)(t_2)| = |\, (Th)(t_1) - (Th)(t_2) \\
&+ (TGTh)(t_1) - (TGTh)(t_2) + \big((TG)^2 Th\big)(t_1) \\
&- \big((TG)^2 Th\big)(t_2) + \ldots + \big((TG)^n Th\big)(t_1) \\
&- \big((TG)^n Th\big)(t_2) + \ldots | \le |(Th)(t_1) \\
&- (Th)(t_2)| + |(TGTh)(t_1) - (TGTh)(t_2)| \\
&+ \big|\big((TG)^2 Th\big)(t_1) - \big((TG)^2 Th\big)(t_2)\big| + \ldots + \\
&\big|\big((TG)^n Th\big)(t_1) - \big((TG)^n Th\big)(t_2)\big| + \ldots = \\
&|(Th)(t_1) - (Th)(t_2)| + |(TG)(Th(t_1) - Th(t_2))| \\
&\big|(TG)^2 (Th(t_1) - Th(t_2))\big| + \ldots + \big|(TG)^n \big(Th(t_1) \\
&- Th(t_2)\big)\big| + \ldots \le |(Th)(t_1) - (Th)(t_2)| \\
&+ \|TG\| \cdot |Th(t_1) - Th(t_2)| + \|TG\|^2 \cdot |Th(t_1) \\
&- Th(t_2)| + \ldots + \|TG\|^n \cdot |Th(t_1) \\
&- Th(t_2)| + \ldots \le (1 + L + \ldots + L^n + \ldots)|(Th)(t_1) \\
&- (Th)(t_2)| = \frac{1}{1-L} |(Th)(t_1) - (Th)(t_2)|.
\end{aligned}$$

Let $h(t) = Fu(t) = \mu u(t) \int_0^1 K(t,s) u(s)\, ds + f(t,u,u'')$. So, by (38), we have

$$\begin{aligned}
&|(Qu)(t_1) - (Qu)(t_2)| = |(HFu)(t_1) - (HFu)(t_2)| \\
&\le \frac{1}{1-L} |(TFu)(t_1) - (TFu)(t_2)| \\
&\le \frac{1}{1-L} (\mu D_1 d^2 + M_d D_3)|t_1 - t_2|.
\end{aligned}$$

We have a similar type of inequality for $|(Qu)''(t_1) - (Qu)''(t_2)|$.

Therefore $Q(V)$ is equicontinuous.

Next we prove that T is continuous. Suppose $u_n, u \in P$ and $\|u_n - u\|_2 \to 0$ which implies that $u_n(t) \to u(t)$, $u_n''(t) \to u''(t)$ uniformly on $[0,1]$. Similarly, for $f(t,u,v) = g(t) \cdot h(t,u,v)$, $h(t, u_n(t), u_n''(t)) \to h(t, u(t), u''(t))$ uniformly on $[0,1]$. The assertion follows from the estimate

$$\begin{aligned}
&|(Hh_2)(t) - (Hh_1)(t)| = |\, (Th_2)(t) - (Th_1)(t) \\
&+ (TGTh_2)(t) - (TGTh_1)(t) + \big((TG)^2 Th_2^n\big)(t) \\
&- \big((TG)^2 Th_1\big)(t) + \ldots + \big((TG)^n Th_2\big)(t) \\
&- \big((TG)^n Th_1\big)(t) + \ldots | \le |(Th_2)(t) \\
&- (Th_1)(t)| + |(TGTh_2)(t) - (TGTh_1)(t)| \\
&+ \big|\big((TG)^2 Th_2\big)(t) - \big((TG)^2 Th_1\big)(t)\big| \\
&+ \ldots + \big|\big((TG)^n Th_2\big)(t) - \big((TG)^n Th_1\big)(t)\big| + \ldots \\
&= |(Th_2)\,(t) - (Th_1)\,(t)| + |(TG)(Th_2(t) - Th_1(t))| \\
&+ \Big|(TG)^2\, (Th_2(t) - Th_1(t))\Big| \\
&+ \ldots + |(TG)^n\, (Th_2(t) - Th_1(t))| + \ldots \\
&\le |(Th_2)\,(t) - (Th_1)\,(t)| + \|TG\| \cdot |Th_2(t) \\
&- Th_1(t)| + \|TG\|^2 \cdot |Th_2(t) - Th_1(t)| + \ldots + \\
&\|TG\|^n \cdot |Th_2(t) - Th_1(t)| + \ldots \\
&\le (1 + L + \ldots + L^n + \ldots) \cdot |Th_2(t) - Th_1(t)| \\
&= \frac{1}{1-L} \cdot |Th_2(t) - Th_1(t)|.
\end{aligned} \tag{39}$$

Let $h_1(t) = Fu_n(t)$ and $h_2(t) = Fu(t)$. So, by (39), we have

$$\begin{aligned}
&|(Qu_n)(t) - (Qu)(t)| = |(HFu_n)(t) - (HFu)(t)| \\
&\le \frac{1}{1-L} |TFu_n(t) - TFu(t)| \\
&\le \mu \frac{1}{1-L} \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, |u_n(s) - u(s)| \\
&\int_0^1 K(s,v)\, |u_n(v) - u(v)|\, dv\, ds\, d\tau \\
&+ \frac{1}{1-L} \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\ |g(s)|\ |h(s, u_n(s), \\
&u_n''(s)) - h(t, u(s), u''(s))| ds\, d\tau,
\end{aligned}$$

and the similar estimate for $|(Qu_n)''(t) - (Qu)''(t)|$ by an application of the standard theorem on the convergence of integrals. Obviously, $Q : P \to P$ is continuous.

The Ascoli-Arzela theorem guarantees that $Q : P \to P$ is completely continuous. ∎

Lemma 10. *If $u(0) = u(1) = 0$ and $u \in C^2[0,1]$, then $\|u\|_0 \le \|u''\|_0$, and so, $\|u\|_2 = \|u''\|_0$.*

Proof. Since $u(0) = u(1)$, there exists $\alpha \in (0,1)$ such that $u'(\alpha) = 0$, and so $u'(t) = \int_\alpha^t u''(s)\, ds$,

$t \in [0,1]$. Hence $|u'(t)| \leq \int_\alpha^t |u''(s)|\,ds \leq \int_0^1 |u''(s)|\,ds \leq \|u''\|_0$, $t \in [0,1]$. Thus $\|u'\|_0 \leq \|u''\|_0$. Since $u(0)=0$, we have $u(t) = \int_0^t u'(s)\,ds$, $t \in [0,1]$, and so $|u(t)| \leq \int_0^1 |u'(s)|\,ds \leq \|u'\|_0$. Thus $\|u\|_0 \leq \|u'\|_0 \leq \|u''\|_0$. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$ and $\|u\|_0 \leq \|u''\|_0$, we obtain that $\|u\|_2 = \|u''\|_0$. ∎

Corollary 1. *Let $r > 0$ and let $u \in \partial B_r \cap P$. Then $\|u\|_2 = \|u''\|_0 = r$.*

Let us denote by $\bar{\mu} = \dfrac{1}{4C_m C_1(1+|\lambda_2|)D_1}$.

3 Main results

In the following we prove two results, in Theorem 1 and Theorem 2, that assert the existence of positive solutions.

Theorem 1. *Suppose that $(H1)$ applies.. Assume that the following condition holds*

$(H2)$
$$\limsup_{w\to 0^+} \frac{f_2(w)}{w} = 0,$$
and
$$\liminf_{|v|\to\infty} \min_{t\in[\frac14,\frac34]} \inf_{u\in[0,+\infty)} \frac{f(t,u,v)}{|v|} = \infty.$$

If $\mu \in (0,\bar{\mu})$, then problem BVP (3) *has at least one positive solution.*

Proof. Let us choose $0 < \widehat{C}_1 \leq \frac{\bar{\mu}}{2}$. Then by $(H2)$, there exists $0 < r < \frac12$ such that

$$f_2(|u|+|v|) \leq \widehat{C}_1(|u|+|v|), 0 \leq |u|+|v| \leq 2r.$$

Let $u \in \partial B_r \cap P$, then by Corollary 1, $\|u\|_2 = \|u''\|_0 = r$ and $u(0) = u(1) = 0$. Also since $\|u\|_0 \leq \|u''\|_0$ we have $|u(t)| \leq \|u\|_0 \leq r$, $|u''(t)| \leq \|u''\|_0 = r$, $\forall t \in [0,1]$, therefore $0 \leq |u(t)| + |u''(t)| \leq 2r$, $\forall t \in [0,1]$.

Thus, by Lemma 2, $(H1)$ and $(H2)$, we have

$$\begin{aligned}
(Qu)(t) &\leq \frac{1}{1-L}(Tu)(t) \\
&\leq \frac{1}{1-L}\Big\{\mu C_1 \int_0^1\int_0^1\int_0^1 G_1(\tau,\tau)G_2(\tau,s) \\
&\quad K(s,v)\,dvdsd\tau\|u\|_0^2 + C_1\int_0^1\int_0^1 G_1(\tau,\tau) \\
&\quad G_2(\tau,s)f_1(s)f_2(|u|+|u''|)\,dsd\tau\} \\
&\leq \frac{C_1}{1-L}\Big\{\mu D_3\|u\|_0^2 + \widehat{C}_1 D_4(\|u\|_0+\|u''\|_0)\Big\} \\
&\leq \mu D_3\frac{C_1}{1-L}\|u\|_0^2 + 2\widehat{C}_1 D_4\|u''\|_0\frac{C_1}{1-L} \\
&\leq \mu D_1\frac{C_1}{1-L}\|u\|_0^2 + 2\widehat{C}_1 D_2\|u''\|_0\frac{C_1}{1-L} \\
&\leq \mu D_1 C_m C_1\|u\|_0^2 + 2\widehat{C}_1 D_2\|u''\|_0 C_m C_1 \\
&\leq \frac14\|u\|_0^2 + \frac14\|u\|_2 \leq \frac14\|u\|_2^2 + \frac14\|u\|_2 \\
&\leq \frac12\|u\|_2,\ \forall u \in \partial B_r\cap P,\ t\in[0,1].
\end{aligned}$$

Consequently,

$$\|Qu\|_0 \leq \frac12\|u\|_2,\ \forall u \in \partial B_r \cap P. \tag{40}$$

Similarly we also have

$$-(Qu)''(t) = (-\lambda_2)(HFu)(t) + (H_1Fu)(t),$$

hence

$$\begin{aligned}
&|(Qu)''(t)| = |\lambda_2||(HFu)(t)| + |(H_1Fu)(t)| \\
&\leq |\lambda_2|\frac{1}{1-L}|(TFu)(t)| + \frac{1}{1-L_1}|(T_1Fu)(t)| \\
&\leq C_m\{\mu C_1|\lambda_2|\int_0^1\int_0^1\int_0^1 G_1(\tau,\tau) \\
&G_2(\tau,s)K(s,v)\,dvdsd\tau\|u\|_0^2 \\
&+ C_1|\lambda_2|\int_0^1\int_0^1 G_1(\tau,\tau)G_2(\tau,s)f_1(s)f_2(|u|+|u''|)\,dsd\tau \\
&+ \mu C_1\int_0^1\int_0^1 G_1(\tau,\tau)K(\tau,v)\,dvd\tau\|u\|_0^2 \\
&+ C_1\int_0^1 G_1(\tau,\tau)f_1(\tau)f_2(|u|+|u''|)\,d\tau\} \\
&\leq C_1C_m\{|\lambda_2|\mu D_3\|u\|_0^2 + \widehat{C}_1 D_4|\lambda_2|(\|u\|_0+\|u''\|_0) \\
&+ \mu D_1\|u\|_0^2 + \widehat{C}_1 D_2(\|u\|_0+\|u''\|_0)\} \\
&\leq C_1C_m\{|\lambda_2|\mu D_1\|u\|_0^2 + \widehat{C}_1 D_2|\lambda_2|\|u''\|_0 \\
&+ \mu D_1\|u\|_0^2 + \widehat{C}_1 D_2 2\|u''\|_0\} \\
&\leq C_1C_m(1+|\lambda_2|)D_1\mu\|u\|_0^2 + C_1C_m(1+|\lambda_2|)D_2\widehat{C}_1 2\|u''\|_0 \\
&\leq \frac14\|u\|_0^2 + \frac14\|u''\|_0 = \frac12\|u\|_2,\ \forall u\in\partial B_r\cap P,\ t\in[0,1].
\end{aligned}$$

Consequently,

$$\|(Qu)''\|_0 \leq \frac12\|u\|_2,\ \forall u \in \partial B_r \cap P. \tag{41}$$

Using (40) and (41) we have

$$\|Qu\|_2 \leq \|Qu\|_0 + \|(Qu)''\|_0 \leq \|u\|_2,\ \forall u \in \partial B_r \cap P. \tag{42}$$

Let us choose $0 < \widehat{C}_2 \leq \frac{1}{\sigma D_5}$. Then, by condition $(H2)$, there exists $R_1 > 0$ such that

$$f(t,u,v) \geq \widehat{C}_2|v|,\ \forall u \in \mathbb{R}_+,\ \forall |v| \geq R_1,\ t \in \left[\frac14,\frac34\right].$$

Let $R > \max\{\frac{R_1}{\sigma}, r\}$. Let $u \in \partial B_R \cap P$, i.e. $\|u''\|_0 = R$. Thus we have

$$\min_{t\in[\frac14,\frac34]} |u''(t)| \geq \sigma\|u''\|_0 = \sigma R > R_1,\ u \in \partial B_R \cap P.$$

Then, by Lemma 1, $(H1)$ and $(H2)$, we have

$$(Qu)\left(\frac{1}{2}\right) \geq (Tu)\left(\frac{1}{2}\right) \geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_2\left(\frac{1}{2}, \tau\right)$$
$$G_1(\tau, s)u(s)K(s, v)u(v)\, dvdsd\tau$$
$$+ \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_2\left(\frac{1}{2}, \tau\right) G_1(\tau, s) f(s, u(s), u''(s))\, dsd\tau$$
$$\geq \widehat{C}_2 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_2\left(\frac{1}{2}, \tau\right) G_1(\tau, s)|u''(s)| dsd\tau$$
$$\geq \widehat{C}_2 \sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_2\left(\frac{1}{2}, \tau\right) G_1(\tau, s)\, dsd\tau \|u''\|_0 \geq \|u''\|_0$$

so

$$(Qu)\left(\frac{1}{2}\right) \geq \|u''\|_0 = \|u\|_2, \ \forall u \in \partial B_R \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Qu\|_0 \leq \|Qu\|_2, \ \forall u \in \partial B_R \cap P.$$

Then, due to Lemma 4, by (42) and the above inequality we see that the problem (3) has at least one positive solution. ∎

Theorem 2. *Suppose that $(H1)$ applies. Assume that the following conditions hold*

$(H3)$

$$\liminf_{|u|+|v| \to 0^+} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, u, v)}{|u| + |v|} = \infty,$$

and

$$\liminf_{|v| \to \infty} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \inf_{u \in [0, +\infty)} \frac{f(t, u, v)}{|v|} = \infty;$$

$(H4)$ *there exists $0 < \varrho < \frac{1}{2}$ such that*

$$\sup_{w \in [0,1]} f_2(w) \leq \frac{\varrho(1 - L)}{4D_2C_1(1 + |\lambda_2|)}. \qquad (43)$$

If $\mu \in (0, \overline{\mu})$, then problem BVP(3) *has at least two positive solutions.*

We note for the argument below that $D_4 \leq D_2$, $\frac{\varrho(1-L)}{4D_2C_1(1+|\lambda_2|)} \leq \frac{\varrho}{4D_2C_1}$, and $\overline{\mu} = \frac{1}{4D_1C_1(1+|\lambda_2|)C_m} \leq \frac{1}{4D_1C_1C_m}$.

Proof. By condition $(H4)$ there exists $0 < \varrho < \frac{1}{2}$ such that (43) is fulfilled. Let $u \in \partial B_\varrho \cap P$, by Corollary 1, $\|u''\|_0 = \varrho$, $u(0) = u(1) = 0$. Also, since $\|u\|_0 \leq \|u''\|_0$ we have $u(t) \leq \|u\|_0 \leq \varrho$, $|u''(t)| \leq \|u''\|_0 = \varrho$, $\forall t \in [0, 1]$, therefore $0 \leq |u(t)| + |u''(t)| < 1$, $\forall t \in [0, 1]$. By condition $(H4)$, $\forall u \in \partial B_\varrho \cap P$ and $t \in [0, 1]$, we have

$$(Qu)(t) \leq \frac{1}{1 - L}(Tu)(t)$$
$$\leq \frac{1}{1 - L}\mu C_1 \int_0^1 \int_0^1 \int_0^1 G_1(\tau, \tau)$$
$$G_2(\tau, s)K(s, v)\, dvdsd\tau \|u\|_0^2 + \frac{1}{1 - L}C_1$$
$$\int_0^1 \int_0^1 G_1(\tau, \tau)G_2(\tau, s)f_1(s)f_2(|u| + |u''|)\, dsd\tau$$
$$\leq \frac{1}{1 - L}C_1\mu D_3\|u\|_0^2 + \frac{\varrho}{4D_2C_1}C_1 \int_0^1 \int_0^1 G_1(\tau, \tau)$$
$$G_2(\tau, s)f_1(s)\, ds \leq \frac{1}{1 - L}C_1\mu D_1\|u\|_0^2$$
$$+ \frac{\varrho}{4D_2} \int_0^1 \int_0^1 G_1(\tau, \tau)G_2(\tau, s)f_1(s)\, ds \leq C_mC_1\mu D_1\|u\|_0^2$$
$$+ \frac{\varrho}{4D_2} \int_0^1 \int_0^1 G_1(\tau, \tau)G_2(\tau, s)f_1(s)\, ds$$
$$\leq \frac{1}{4}\|u\|_0^2 + \frac{1}{4}\varrho = \frac{1}{4}\|u\|_0^2 + \frac{1}{4}\|u''\|_0 = \frac{1}{4}\|u\|_0^2 + \frac{1}{4}\|u\|_2$$
$$\leq \frac{1}{4}\|u\|_2^2 + \frac{1}{4}\|u\|_2 \leq \frac{1}{2}\|u\|_2, \ \forall u \in \partial B_\varrho \cap P, \ t \in [0, 1].$$

Consequently, we get

$$\|Qu\|_0 \leq \frac{1}{2}\|u\|_2, \ \forall u \in \partial B_\varrho \cap P. \qquad (44)$$

Similarly we also have

$$-(Qu)''(t) = (-\lambda_2)(HFu)(t) + (H_1Fu)(t),$$

hence

$$|(Qu)''(t)| = |\lambda_2||(HFu)(t)| + |(H_1Fu)(t)|$$
$$\leq |\lambda_2|\frac{1}{1 - L}|(TFu)(t)| + \frac{1}{1 - L_1}|(T_1Fu)(t)|$$
$$\leq C_m\{\mu C_1|\lambda_2| \int_0^1 \int_0^1 \int_0^1 G_1(\tau, \tau)$$
$$G_2(\tau, s)K(s, v)\, dvdsd\tau \|u\|_0^2$$
$$+ |\lambda_2|C_1 \int_0^1 \int_0^1 G_1(\tau, \tau)$$
$$G_2(\tau, s)f_1(s)f_2(|u| + |u''|)\, dsd\tau$$
$$+ \mu C_1 \int_0^1 \int_0^1 G_1(\tau, \tau)K(\tau, v)\, dvd\tau \|u\|_0^2$$
$$+ C_1 \int_0^1 G_1(\tau, \tau)f_1(\tau)f_2(|u| + |u''|)\, d\tau\}$$
$$\leq C_m\{\mu C_1|\lambda_2|D_3\|u\|_0^2$$
$$+ |\lambda_2|C_1\frac{\varrho(1 - L)}{4D_2}\frac{1}{C_1(1 + |\lambda_2|)}$$
$$\int_0^1 \int_0^1 G_1(\tau, \tau)G_2(\tau, s)f_1(s)\, dsd\tau$$

$$+\mu C_1 D_1\|u\|_0^2 + C_1\frac{\varrho(1-L)}{4D_2}\frac{1}{C_1(1+|\lambda_2|)}$$
$$\int_0^1 G_1(\tau,\tau)f_1(\tau)\,d\tau\}$$
$$\leq C_m\mu C_1 D_1(1+|\lambda_2|)\|u\|_0^2$$
$$+\frac{1}{4}(1+|\lambda_2|)\frac{1}{(1+|\lambda_2|)}C_m C_1\varrho(1-L)$$
$$\leq C_m\mu C_1 D_1(1+|\lambda_2|)\|u\|_0^2$$
$$+\frac{1}{4}(1+|\lambda_2|)\frac{1}{(1+|\lambda_2|)}C_1\varrho$$
$$\leq \frac{1}{4}\|u\|_0^2+\frac{1}{4}\varrho=\frac{1}{4}\|u\|_0^2+\frac{1}{4}\|u\|_2$$
$$\leq \frac{1}{4}\|u\|_2^2+\frac{1}{4}\|u\|_2\leq\frac{1}{2}\|u\|_2,$$
$$\forall u\in\partial B_\varrho\cap P,\ t\in[0,1].$$

Consequently,

$$\|(Qu)''\|_0\leq\frac{1}{2}\|u\|_2,\ \forall u\in\partial B_\rho\cap P. \tag{45}$$

Using (44) and (45) we have

$$\|Qu\|_2\leq\|Qu\|_0+\|\big(Qu\big)''\|_0\leq\|u\|_2,\ \forall u\in\partial B_\rho\cap P. \tag{46}$$

Let us choose $0<c_3\leq\frac{1}{\sigma D_5}$. Then, by condition $(H3)$, there exists $0<r<\varrho$ such that

$$f(t,u,v)\geq c_3(|u|+|v|),\ \ \forall u\in[0,r]$$
$$\forall\,|v|\in[0,r],\ \ t\in\left[\frac{1}{4},\frac{3}{4}\right].$$

Let $u\in\partial B_r\cap P$, by Corollary 1, $\|u''\|_0=r$, $u(0)=u(1)=0$. Also, since $\|u\|_0\leq\|u''\|_0$ we have

$$0\leq u(t)\leq\|u\|_0\leq r,$$
$$0\leq|u''(t)|\leq\|u''\|_0=\|u\|_2=r,$$
$$\forall\,u\in\partial B_r\cap P.$$

Also we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}|u''(t)|\geq\sigma\|u''\|_0=\sigma r,\ \forall u\in\partial B_r\cap P.$$

The estimate for $(Tu)\left(\frac{1}{2}\right)$ is similar to that in the proof of Theorem 1, i.e. from Lemma 1 and $(H1)$ we have

$$(Qu)\left(\frac{1}{2}\right)\geq(Tu)\left(\frac{1}{2}\right)c_3\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_2\left(\frac{1}{2},\tau\right)$$
$$\geq G_1(\tau,s)(|u(s)|+|u''(s)|)\,dsd\tau\geq c_3\sigma$$
$$\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_2\left(\frac{1}{2},\tau\right)G_1(\tau,s)\,dsd\tau\|u''\|_0\geq\|u''\|_0.$$

Thus

$$(Qu)\left(\frac{1}{2}\right)\geq\|u''\|_0=\|u\|_2,\ \forall u\in\partial B_r\cap P.$$

Consequently,

$$\|u\|_2\leq\|Qu\|_0\leq\|Qu\|_2,\ \forall u\in\partial B_r\cap P.$$

Finally we show that for sufficiently large $R>\frac{1}{2}$, it holds

$$\|Qu\|_2\geq\|u\|_2,\ \forall u\in\partial B_R\cap P.$$

To see this we choose $0<c_2\leq\frac{1}{\sigma D_5}$. Due to condition $(H4)$, there exists $R_1>0$ such that

$$f(t,u,v)\geq c_2|v|,\ \forall u\in\mathbb{R}_+,\ \forall|v|\geq R_1,\ t\in\left[\frac{1}{4},\frac{3}{4}\right].$$

Let $R>\max\{\frac{R_1}{\sigma},\frac{1}{2}\}$. Let $u\in\partial B_R\cap P$, by Corollary 1, $\|u''\|_0=R$. Thus we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}|u''|\geq\sigma\|u''\|_0=\sigma R>R_1,\ \forall u\in\partial B_R\cap P.$$

Then, by Lemma 1, $(H1)$ and $(H4)$, we have

$$(Qu)\left(\frac{1}{2}\right)\geq(Tu)\left(\frac{1}{2}\right)\geq c_2\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_2\left(\frac{1}{2},\tau\right)$$
$$G_1(\tau,s)|u''(s)|dsd\tau\geq c_2\sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_2\left(\frac{1}{2},\tau\right)$$
$$G_1(\tau,s)\,dsd\tau\|u''\|_0\geq\|u''\|_0$$

so

$$(Qu)\left(\frac{1}{2}\right)\geq\|u''\|_0=\|u\|_2,\ \forall u\in\partial B_R\cap P.$$

Consequently,

$$\|u\|_2\leq\|Qu\|_0\leq\|Qu\|_2,\ \forall u\in\partial B_R\cap P.$$

Then by Lemma 4, we know that Q has at least two fixed points in $\big(\overline{B}_R\backslash B_\rho\big)\cap P$ and $\big(\overline{B}_\rho\backslash B_r\big)\cap P$, i.e. problem (3) has at least two positive solutions. ∎

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