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THE COMPLEX VERSION OF A RESULT FOR REAL ITERATIVE FUNCTIONS

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Abstract

The purpose of this paper is to show a complex version for complex iterative functions of a result for real iterative functions and to give some applications for complex nonlinear equations.

Keywords: complex nonlinear equations, complex iterative functions, Banach fixed point theorem, Lagrange mean value theorem in complex case

1 Introduction

It is known the following result for iterative functions on the real line, see for example [1] or [2]:

Theorem 1. *(general theorem for real iterative functions) If φ is derivable on the interval $J = [x_0 - \delta, x_0 + \delta]$, $\delta > 0$ and the derivative function φ' satisfies the inequality $0 \le |\varphi'(x)| \le m < 1$ for every $x \in J$ and the point $x_1 = \varphi(x_0)$ verifies the inequality $|x_1 - x_0| \le (1 - m)\delta$, then:*

- *we can form the sequence $\{x_k\}_{k\in\mathbb{N}}$ with the iterative rule $x_{k+1} = \varphi(x_k)$, $k \in \mathbb{N}$, such that for every $k \in \mathbb{N}$ we have $x_k \in J$;*
- *there exists $\lim_{k\to\infty} x_k = x^* \in J$;*
- *x^* is the unique solution of the equation $\varphi(x) = x$ in the interval J.*

The purpose of this paper is to show a complex variant of this theorem.

2 Main part

Let us consider the closed disc $B(w, r) = \{z \in \mathbb{C}/|z - w| \le r\}$ in the complex plane $\mathbb{C}$, with center w and radius $r > 0$. First we remember the Banach fixed point theorem in the case of the closed disc $B(w, r)$:

Theorem 2. *Let $\phi : B(w, r) \to B(w, r)$ be a contraction, i.e. there exists the constant $\alpha \in [0, 1)$ such that $|\phi(z) - \phi(v)| \le \alpha \cdot |z - v|$ for every $z, v \in B(w, r)$. Then the function ϕ has a unique fixed point in $B(w, r)$, which can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$ given by the iteration $z_{k+1} = \phi(z_k), k \in \mathbb{N}$, for every $z_0 \in B(w, r)$.*

Proof. Because $B(w, r) \subset \mathbb{C}$ is a closed disc in the complex plane $\mathbb{C}$, will be a Banach space, too. Now we apply the Banach fixed point theorem for the function $\phi : B(w, r) \to B(w, r)$. □

We say that the function $\phi : B(w, r) \to \mathbb{C}$ is holomorphic function on the closed disc $B(w, r)$, if it is complex derivable in every complex point $z \in B(w, r)$. If the point z is a boundary point of the closed disc $B(w, r)$, then we suppose that the complex function ϕ is defined on a small open disc with center z and it is complex derivable in z.

Theorem 3. *Let $\phi : B(w, r) \to \mathbb{C}$ be a holomorphic function on the closed disc $B(w, r)$, and $z, v \in B(w, r)$ two distinct points. Then there exist points $u, s \in [z, v] = \{t \cdot z + (1 - t) \cdot v/t \in [0, 1]\}$ (the line segment $[z, v] \subset \mathbb{C}$ with endpoints z and v) such that $Re\phi'(u) = Re(\frac{\phi(z)-\phi(v)}{z-v})$ and $Im\phi'(s) = Im(\frac{\phi(z)-\phi(v)}{z-v})$, where $Re()$ is the real part and $Im()$ is the imaginary part of the complex function ϕ.*

Proof. See for example [3] or [4]. □

Theorem 4. *(general theorem for complex iterative functions) If $\phi : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on the closed disc $B(z_0, r), z_0 \in \mathbb{C}, r > 0$, such that the derivative function ϕ' satisfies the inequality $0 \leq |\phi'(z)| \leq m < \frac{\sqrt{2}}{2}$ for every $z \in B(z_0, r)$ and the point $z_1 = \phi(z_0)$ verifies the inequality $|z_1 - z_0| \leq (1 - \sqrt{2} \cdot m) \cdot r$, then:*

- *we can form the sequence $\{z_k\}_{k\in\mathbb{N}}$ with the iterative rule $z_{k+1} = \phi(z_k)$, $k \in \mathbb{N}$, such that for every $k \in \mathbb{N}$ we have $z_k \in B(z_0, r)$;*
- *there exists the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$ and $\lim_{k\to\infty} z_k = z^* \in B(z_0, r)$;*
- *z^* is the unique solution of the equation $\phi(z) = z$ in the closed disc $B(z_0, r)$.*

Proof. Using theorem 3 for $z \in B(z_0, r)$ and $z \neq z_0$ we get: $|\frac{\phi(z)-\phi(z_0)}{z-z_0}|^2 = Re^2(\frac{\phi(z)-\phi(z_0)}{z-z_0}) + Im^2(\frac{\phi(z)-\phi(z_0)}{z-z_0}) = Re^2(\phi'(u)) + Im^2(\phi'(s)) \leq Re^2(\phi'(u)) + Im^2(\phi'(u)) + Re^2(\phi'(s)) + Im^2(\phi'(s)) = |\phi'(u)|^2 + |\phi'(s)|^2 \leq m^2 + m^2 = 2 \cdot m^2$, so $|\phi(z) - \phi(z_0)| \leq \sqrt{2} \cdot m \cdot |z - z_0|$. First we show that $\phi(B(z_0, r)) \subset B(z_0, r)$, i.e. $\phi : B(z_0, r) \to B(z_0, r)$. Indeed, for every $z \in B(z_0, r)$ we obtain: $|\phi(z) - z_0| = |\phi(z) - \phi(z_0) + z_1 - z_0| \leq |\phi(z)-\phi(z_0)|+|z_1-z_0| \leq \sqrt{2}\cdot m\cdot|z-z_0|+|z_1-z_0| \leq \sqrt{2} \cdot m \cdot r + (1 - \sqrt{2} \cdot m) \cdot r = r$. Using again theorem 3 for every $z, v \in B(z_0, r), z \neq v$ results $|\phi(z) - \phi(v)| \leq \sqrt{2} \cdot m \cdot |z - v|$. We choose $\alpha = \sqrt{2} \cdot m < 1$ and the complex function $\phi : B(z_0, r) \to B(z_0, r)$ verifies the conditions of theorem 2. Consequently we can deduce the statements of theorem 4. q.e.d. □

Observation 1. *Because ϕ' is a holomorphic function using the maximum modulus principle it is enough to have in theorem 4 the following condition: $0 \leq |\phi'(z)| \leq m < \frac{\sqrt{2}}{2}$ for every $z \in C(z_0, r) = \{z \in \mathbb{C}/|z - z_0| = r\}$.*

3 Discussion and conclusion

Next we give some applications for theorem 4.

Conclusion 1. *Using the translation the complex equation $f(z) = 0$ is equivalent with the complex equation $z + f(z) = z$. We can consider the complex iterative function $\phi(z) = z + f(z)$. Now we apply theorem 4 for the iterative function ϕ and we obtain the following result for the function f: if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$ and $r > 0$, $|1 + f'(z)| \leq m < \frac{\sqrt{2}}{2}$ for every $z \in B(z_0, r)$, and $|f(z_0)| \leq (1 - \sqrt{2} \cdot m) \cdot r$, then the complex equation $f(z) = 0$ has a unique solution z^* in the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$, given by the iterative formula $z_{k+1} = z_k + f(z_k), k \in \mathbb{N}$.*

Conclusion 2. *Using the translation and the homothety the complex equation $f(z) = 0$ is equivalent with the complex equation $z + \omega \cdot f(z) = z, \omega \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We can consider the complex iterative function $\phi(z) = z + \omega \cdot f(z), \omega \in \mathbb{C}^*$. Now we apply theorem 4 for the iterative function ϕ and we obtain the following result for the function f: if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$ and $r > 0$, $|1 + \omega \cdot f'(z)| \leq m < \frac{\sqrt{2}}{2}$ for every $z \in B(z_0, r)$, and $|f(z_0)| \leq (1-\sqrt{2}\cdot m)\cdot \frac{r}{|\omega|}$, then the complex equation $f(z) = 0$ has a unique solution z^* in the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$, given by the iterative formula $z_{k+1} = z_k + \omega \cdot f(z_k), k \in \mathbb{N}$.*

Conclusion 3. *Using the complex Newton's transformation the complex equation $f(z) = 0$ is equivalent with the complex equation $z - \frac{f(z)}{f'(z)} = z$, where we suppose that $f'(z) \neq 0$. We can consider the complex iterative function $\phi(z) = z - \frac{f(z)}{f'(z)}$, the complex variant of the Newton's method. Now we apply theorem 4 for the iterative function ϕ and we obtain the following result for the function f: if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$ and $r > 0$, and $f'(z) \neq 0$ for every $z \in B(z_0, r)$, and $\frac{|f(z)\cdot f''(z)|}{[f'(z)]^2} \leq m < \frac{\sqrt{2}}{2}$ for every $z \in B(z_0, r)$, and $\left|\frac{f(z_0)}{f'(z_0)}\right| \leq (1 - \sqrt{2} \cdot m) \cdot r$, then the complex equation $f(z) = 0$ has a unique solution z^* in the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$, given by the complex Newton's iterative formula $z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}, k \in \mathbb{N}$.*

Conclusion 4. *Using the transformation of complex parallel method the complex equation $f(z) = 0$ is equivalent with the complex equation $z - \frac{1}{\lambda} \cdot f(z) = z, \lambda \in \mathbb{C}^*$. We can consider the complex iterative function $\phi(z) = z - \frac{1}{\lambda} \cdot f(z), \lambda \in \mathbb{C}^*$, the complex variant of the parallel method. Now we apply theorem 4 for the iterative function ϕ and we obtain the following result for the function f: if the complex function $f : B(z_0, r) \to \mathbb{C}$ is a holomorphic function on $B(z_0, r)$, with $z_0 \in \mathbb{C}$ and $r > 0$, $|1 - \frac{1}{\lambda} \cdot f'(z)| \leq m < \frac{\sqrt{2}}{2}$ for every $z \in B(z_0, r)$, and $|f(z_0)| \leq (1-\sqrt{2}\cdot m)\cdot r\cdot|\lambda|$, then the complex equation $f(z) = 0$ has a unique solution z^* in the closed disc $B(z_0, r) \subset \mathbb{C}$ and z^* can be obtained as the limit of the sequence $\{z_k\}_{k\in\mathbb{N}}$, given by the iterative formula of the complex parallel method $z_{k+1} = z_k - \frac{1}{\lambda} \cdot f(z_k), k \in \mathbb{N}$.*

Conclusion 5. *Using the transformation of complex chord method, the complex equation $f(z) = 0$ is equivalent with the complex equation $z - f(z) \cdot \frac{z-a}{f(z)-f(a)} = z$, where $a \in \mathbb{C}$ is a fixed complex number, $z \neq a$ and $f(z) \neq f(a)$ for $z \neq a$. We can consider the complex iterative function $\phi(z) =$*

$z - f(z) \cdot \frac{z-a}{f(z)-f(a)}$, *where* $a, z \in \mathbb{C}$, $z \neq a$, *and* $f(z) \neq f(a)$ *for* $z \neq a$, *the complex variant of the chord method. The condition* $f(a) \neq 0$ *implies for* $z \neq a$ *that* $\phi(z) \neq a$, *too. Indeed, from equality* $\phi(z) = a$ *we get* $z - f(z) \cdot \frac{z-a}{f(z)-f(a)} = a$, *which means that* $(z-a) \cdot \frac{f(a)}{f(z)-f(a)} = 0$. *Consequently, from the condition* $f(a) \neq 0$ *we can deduce* $z = a$.*Now we apply theorem 4 for the iterative function* ϕ *and we obtain the following result for the function f: if the complex function* $f : B(z_0, r) \to \mathbb{C}$ *is a holomorphic function on* $B(z_0, r)$, *with* $z_0 \in \mathbb{C}$ *and* $r > 0$, *and* $a \in B(z_0, r)$ *is a fixed complex number such that* $f(a) \neq 0$ *and for every* $z \in B(z_0, r) - \{a\}$ *we have* $f(z) \neq f(a)$, $\frac{|f(a)|\,|f(a)-f(z)+f'(z)\cdot(z-a)|}{|f(z)-f(a)|^2} \leq m < \frac{\sqrt{2}}{2}$ *for every* $z \in B(z_0, r) - \{a\}$, *and* $\left|\frac{(z_0-a)f(z_0)}{f(z_0)-f(a)}\right| \leq (1-\sqrt{2}\cdot m)\cdot r$ *with* $z_0 \neq a$, *then the complex equation* $f(z) = 0$ *has a unique solution* z^* *in the closed disc* $B(z_0, r) \subset \mathbb{C}$ *and* z^* *can be obtained as the limit of the sequence* $\{z_k\}_{k\in\mathbb{N}}$, *given by the iterative formula of the complex chord method* $z_{k+1} = z_k - f(z_k) \cdot \frac{z_k - a}{f(z_k)-f(a)}, k \in \mathbb{N}$.

Conclusion 6. *Using the transformation of complex Steffensen's method, the complex equation* $f(z) = 0$ *is equivalent with the complex equation* $z - \frac{f^2(z)}{f(z+f(z))-f(z)} = z$. *We can consider the complex iterative function* $\phi(z) = z - \frac{f^2(z)}{f(z+f(z))-f(z)}$, *the complex variant of the Steffensen's method. Now we apply theorem 4 for the iterative function* ϕ *and we obtain the following result for the function f: if the complex function* $f : B(z_0, r) \to \mathbb{C}$ *is a holomorphic function on* $B(z_0, r)$, *with* $z_0 \in \mathbb{C}$ *and* $r > 0$,

$$\begin{aligned} & |1 - \{2f(z)\cdot f'(z)[f(z+f(z)) - f(z)] \\ - \; & f^2(z)\cdot[f'(z+f(z))\cdot(1+f'(z)) - f'(z)]\} \\ : \; & [f(z+f(z)) - f(z)]^2| \\ = \; & |1 - \{f^2(z)\cdot f'(z) + 2f(z)\cdot f'(z)\cdot f(z+f(z)) \\ - \; & f^2(z)\cdot f'(z+f(z))\cdot(1+f'(z))\} \\ : \; & [f(z+f(z)) - f(z)]^2| \leq m < \frac{\sqrt{2}}{2} \end{aligned}$$

for every $z \in B(z_0, r)$, *and* $\left|\frac{f^2(z_0)}{f(z_0+f(z_0))-f(z_0)}\right| \leq (1-\sqrt{2}\cdot m)\cdot r$, *then the complex equation* $f(z) = 0$ *has a unique solution* z^* *in the closed disc* $B(z_0, r) \subset \mathbb{C}$ *and* z^* *can be obtained as the limit of the sequence* $\{z_k\}_{k\in\mathbb{N}}$, *given by the iterative formula of the complex Steffensen's method* $z_{k+1} = z_k - \frac{f^2(z_k)}{f(z_k+f(z_k))-f(z_k)}, k \in \mathbb{N}$.

Conclusion 7. *The equation* $z^n - a = 0$, *where* $a \in \mathbb{C}^*$ *is a fixed complex number and* $n \in \mathbb{N}, n \geq 2$ *is a fixed natural number, is equivalent with the complex equation* $\frac{1}{2}\cdot(z + \frac{a}{z^{n-1}}) = z, z \in \mathbb{C}^*$. *We can consider the complex iterative function* $\phi(z) = \frac{1}{2}\cdot(z + \frac{a}{z^{n-1}})$. *Now we apply theorem 4 for the iterative function* ϕ *and we obtain the following result: if we fix* $z_0 \in \mathbb{C}$ *and* $r > 0$ *such that* $0 \notin B(z_0, r)$ *and* $|1 - \frac{(n-1)a}{z^n}| \leq 2\cdot m < \sqrt{2}$ *for every* $z \in B(z_0, r)$, *and* $|z_0 - \frac{a}{z_0^{n-1}}| \leq 2\cdot(1-\sqrt{2}\cdot m)\cdot r$, *then the complex sequence* $\{z_k\}_{k\in\mathbb{N}}$, *given by the iterative formula* $z_{k+1} = \frac{1}{2}\cdot(z_k + \frac{a}{z_k^{n-1}})$ *is convergent and tends to* $\sqrt[n]{a}$, *the complex nth root of* $a \in \mathbb{C}^*$. *We mention the particular case for* $n = 2$: *if we fix* $z_0 \in \mathbb{C}$ *and* $r > 0$ *such that* $0 \notin B(z_0, r)$, *and* $|1 - \frac{a}{z^2}| \leq 2\cdot m < \sqrt{2}$ *for every* $z \in B(z_0, r)$, *and* $|z_0 - \frac{a}{z_0}| \leq 2\cdot(1-\sqrt{2}\cdot m)\cdot r$, *then the complex sequence* $\{z_k\}_{k\in\mathbb{N}}$, *given by the iterative formula* $z_{k+1} = \frac{1}{2}\cdot(z_k + \frac{a}{z_k})$ *is convergent and tends to* $\sqrt{a}$, *the complex square root of* $a \in \mathbb{C}^*$.

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