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EXISTENCE AND UNIQUENESS OF THE INFINITE MATRIX FACTORIZATION LU

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Abstract

The purpose of this paper is to give conditions for the existence and uniqueness of the infinite matrix factorization LU.

Keywords: infinite matrix, operations with infinite matrices, LU matrix factorization, infinite lower triangular matrix, infinite upper triangular matrix

1 Introduction

First of all we denote the set $\mathbb{N}-\{0\}$ with $\mathbb{N}^*$. Next we give some definitions. A is called a real (complex) infinite matrix, if has infinite, but numerable rows and columns, i.e. $A=(a_{ij})_{i,j\in\mathbb{N}^*}$, where the elements $a_{ij}\in\mathbb{R}$ are real numbers ($a_{ij}\in\mathbb{C}$ are complex numbers) for every $i,j\in\mathbb{N}^*$. The infinite matrices A and $B=(b_{ij})_{i,j\in\mathbb{N}^*}$ are equal, i.e. $A=B$, if $a_{ij}=b_{ij}$ for every $i,j\in\mathbb{N}^*$. If A and B are two real (complex) infinite matrices, then we can define the sum of these matrices: $A+B=(a_{ij}+b_{ij})_{i,j\in\mathbb{N}^*}$ and multiplication by real (complex) scalars: $\alpha\cdot A=(\alpha\cdot a_{ij})_{i,j\in\mathbb{N}^*}$, where $\alpha\in\mathbb{R}$ ($\alpha\in\mathbb{C}$). The product of these matrices we define like $A\cdot B=(\sum_{k=1}^{\infty}a_{ik}\cdot b_{kj})_{i,j\in\mathbb{N}^*}$, where the product matrix exists if and only if all the series $\sum_{k=1}^{\infty}a_{ik}\cdot b_{kj}$ are convergent series for every $i,j\in\mathbb{N}^*$. In other case the product matrix does not exist.

2 Main part

The LU factorization for finite matrices was introduced by Tadeusz Banachiewicz in 1938, see for example [1]. We extended the LU factorization from finite matrices to infinite matrices in [2]. The infinite matrix $L=(l_{ij})_{i,j\in\mathbb{N}^*}$ we call infinite lower triangular matrix, if $l_{ij}=0$ for every $i,j\in\mathbb{N}^*$ and $i<j$. The infinite matrix $U=(u_{ij})_{i,j\in\mathbb{N}^*}$ we call infinite upper triangular matrix, if $u_{ij}=0$ for every $i,j\in\mathbb{N}^*$ and $i>j$. The next result we showed in [2].

Proposition 1. *For the infinite matrix A we can obtain the LU infinite matrix factorization, i.e. there exist L infinite lower triangular matrix and U infinite upper triangular matrix, such that $A=L\cdot U$. If we choose the elements of the principal diagonal of L equals one, i.e. $l_{kk}=1$ for every $k\in\mathbb{N}^*$, then the LU infinite matrix factorization is unique determined.*

Proof. We use the mathematical induction method. First we determine the first column of the infinite matrix L and the first row of the infinite matrix U using the following relations obtained by matrix multiplication of the rows of L with the columns of U: $1\cdot u_{11}=a_{11}$ and for every $i\in\mathbb{N}$, $i\geq 2$ we have $l_{i1}\cdot u_{11}=a_{i1}$. At the same time for every $j\in\mathbb{N}$, $j\geq 2$ we get $1\cdot u_{1j}=a_{1j}$. We suppose that $a_{11}\neq 0$, so $u_{11}=a_{11}$ with $u_{11}\neq 0$, for $i\geq 2$ $l_{i1}=\frac{a_{i1}}{u_{11}}$ and for $j\geq 2$ $u_{1j}=a_{1j}$.

Next we suppose that we calculated the elements of L from the first $n-1$ columns and the elements of U from the first $n-1$ rows with $n\geq 2$. By the mathematical induction step next we determine the elements of L from the column n: l_{in} for $i\geq n+1$ and the elements of U from the row n: u_{nj} for $j\geq n$ using the following relations obtained with matrix multiplication: $\sum_{k=1}^{n-1}l_{nk}\cdot u_{kn}+1\cdot u_{nn}=a_{nn}$, so $u_{nn}=a_{nn}-\sum_{k=1}^{n-1}l_{nk}\cdot u_{kn}$. We suppose that $u_{nn}\neq 0$. For $i\geq n+1$ we have $\sum_{k=1}^{n}l_{ik}\cdot u_{kn}=a_{in}$, so

$l_{in} = \frac{a_{in} - \sum_{k=1}^{n-1} l_{ik} \cdot u_{kn}}{u_{nn}}$. For $j \geq n+1$ we have $\sum_{k=1}^{n} l_{nk} \cdot u_{kj} = a_{nj}$, so $u_{nj} = a_{nj} - \sum_{k=1}^{n-1} l_{nk} \cdot u_{kj}$. □

Observation 1. *At the same time we can choose the elements of the principal diagonal of U equals one, i.e.* $u_{kk} = 1$ *for every* $k \in \mathbb{N}^*$, *then the LU infinite matrix factorization is also unique determined.*

In the above proof we can see that the necessary and sufficient conditions for the existence and uniqueness of the infinite matrix factorization LU there are the conditions $u_{nn} \neq 0$ for every $n \in \mathbb{N}^*$.

We denote for every $n \in \mathbb{N}^*$ by

$$A_n = \begin{pmatrix} a_{11} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix},$$

$$L_n = \begin{pmatrix} l_{11} & & & 0 \\ l_{21} & l_{22} & & \\ \vdots & & \ddots & \\ l_{n,1} & l_{n,2} & \dots & l_{n,n} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1,n} \\ & u_{22} & \dots & u_{2,n} \\ & & \ddots & \vdots \\ 0 & & & u_{n,n} \end{pmatrix}.$$

Proposition 2. *There exists and it is unique the infinite matrix factorization LU if and only if* $det(A_n) \neq 0$ *for every* $n \in \mathbb{N}^*$.

Proof. We use the mathematical induction method. Because $det(A_1) = a_{11}$, then from the infinite matrix equality $A = LU$ we obtain $A_1 = L_1 \cdot U_1$, i.e. $a_{11} = l_{11} \cdot u_{11} = 1 \cdot u_{11} = u_{11}$. So $det(A_1) \neq 0$ if and only if $u_{11} \neq 0$. We suppose that $u_{11} \neq 0, u_{22} \neq 0, \dots, u_{n-1,n-1} \neq 0$. From the infinite matrix equality $A = LU$ cutting the first n rows and the first n columns we deduce $A_n = L_n \cdot U_n$. The square, invertible, finite matrix A_n admits an $L_n \cdot U_n$ factorization if and only if all its leading principal minors are nonzero. The $L_n \cdot U_n$ factorization is unique if we require that the diagonal of L_n (or U_n) consists of ones, see for example [3]. In our case $det(A_k) \neq 0$ for every $k = \overline{1,n}$. The equality $A_n = L_n \cdot U_n$ implies $det(A_n) = det(L_n \cdot U_n) = det(L_n) \cdot det(U_n) = 1 \cdot u_{11} \cdot u_{22} \cdot \dots \cdot u_{n-1,n-1} \cdot u_{n,n}$. The condition $det(A_n) \neq 0$ is equivalent with $u_{n,n} \neq 0$. This ends our proof. □

Observation 2. *It is necessary to put the conditions* $det(A_n) \neq 0$ *for every* $n \in \mathbb{N}^*$. *Indeed, for example if* $det(A_1) = 0$ *and* $det(A_n) \neq 0$ *for every* $n \geq 2$, *then we can not realize the infinite matrix factorization LU. We have* $a_{11} = 0$ *and* $a_{11} \cdot a_{22} - a_{12} \cdot a_{21} \neq 0$, *so* $a_{11} = 0$ *and* $a_{12} \neq 0$ *and* $a_{21} \neq 0$. *From the equality* $A = LU$ *we obtain* $A_1 = L_1 \cdot U_1$ *and we get* $a_{11} = l_{11} \cdot u_{11} = 1 \cdot u_{11} = u_{11}$, *i.e.* $u_{11} = 0$. *Next from the equality* $A = LU$ *we deduce* $A_2 = L_2 \cdot U_2$, *so* $l_{21} \cdot u_{11} = a_{21}$. *This implies* $a_{21} = l_{21} \cdot u_{11} = l_{21} \cdot 0 = 0$, *which means a contradiction.*

Observation 3. *We mention that the above presented algorithm is true also for finite matrices and we can obtain the inverse matrix for a given invertible finite matrix, too.*

3 Discussion and conclusion

At the end we show two concrete examples.

Example 1. *Let A be an infinite matrix with elements given in the following way:* $a_{11} = 2$, *for* $k \geq 2$ $a_{kk} = 1$, *for* $k \geq 1$ $a_{k,k+1} = 1$, *for* $k \geq 2$ $a_{k,k-1} = -2$, *and the other elements are equal zero. We verify the conditions of proposition 2. We have* $det(A_1) = 2, det(A_2) = 4, det(A_3) = 8$. *For* $n \geq 4$ *we calculate* $det(A_n)$ *taking the Laplace expansion along the last row, the row number* n, *after we make the Laplace expansion along the last column, column number* $n-1$, *of the cofactor corresponding to the element* -2. *In this way we get the recurrence relation* $det(A_n) = det(A_{n-1}) + 2 \cdot det(A_{n-2})$ *and we deduce that* $det(A_n) = 2^n \neq 0$. *Using proposition 1 and proposition 2 we obtain the infinite matrices L and U with elements: for* $k \geq 1$ $l_{kk} = 1$, *for* $k \geq 2$ $l_{k,k-1} = -1$, *and the other elements are zero, for* $k \geq 1$ $u_{kk} = 2$, *for* $k \geq 1$ $u_{k,k+1} = 1$, *and the other elements are zero.*

Example 2. *Let A be an infinite matrix with elements given in the following way:* $a_{11} = 1$, *for* $k \geq 2$ $a_{kk} = 5$, *for* $k \geq 1$ $a_{k,k+1} = 2$, *for* $k \geq 2$ $a_{k,k-1} = 2$, *and the other elements are equal zero. We verify the conditions of proposition 2. We have* $det(A_1) = 1, det(A_2) = 1, det(A_3) = 1$. *For* $n \geq 4$ *we calculate* $det(A_n)$ *taking the Laplace expansion along the last row, the row number* n, *after we make the Laplace expansion along the last column, column number* $n-1$, *of the cofactor corresponding to the element* 2. *In this way we get the recurrence relation* $det(A_n) = 5 \cdot det(A_{n-1}) - 4 \cdot det(A_{n-2})$ *and we deduce that* $det(A_n) = 1 \neq 0$. *Using proposition 1 and proposition 2 we obtain the infinite matrices L and U with elements: for* $k \geq 1$ $l_{kk} = 1$, *for* $k \geq 2$ $l_{k,k-1} = 2$, *and the other elements are zero, for* $k \geq 1$ $u_{kk} = 1$, *for* $k \geq 1$ $u_{k,k+1} = 2$, *and the other elements are zero.*

In [4] we used the infinite matrix factorization LU to obtain an algorithm in order to calculate the inverse matrix of an infinite matrix.

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