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INTEGRAL FORM OF AN ALGEBRAIC INEQUALITY

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Abstract

The purpose of this paper is to show the integral form of an algebraic inequality and to give some applications.

Keywords: algebraic inequality, Riemann integral

1 Introduction

It is well known the original Schur inequality with four variables:

$$x^t\cdot(x-y)\cdot(x-z)+y^t\cdot(y-x)(y-z)+z^t(z-x)(z-y)\geq 0, \tag{1}$$

where $x, y, z, t \geq 0$ are positive real numbers [1], [2].

In this paper we will consider a Schur type inequality with three variables:

$$x^t\cdot(x-y)+y^t\cdot(y-x)\geq 0,$$

where $x, y, t \geq 0$ are positive real numbers. The verification is immediately

$$(x^t-y^t)\cdot(x-y)\geq 0.$$

Because the inequality is symmetric in the variables x, y without restriction of the generality we can suppose $x \geq y \geq 0$ so $x^t \geq y^t \geq 0$.

The main parts of these results was presented at [3].

2 Main part

Let $f, g : [a, b] \to [0, +\infty)$ be two continuous functions, where $a, b \in \mathbb{R}, a < b$. For every $v \in [a, b]$ we denote $x = f(v) \geq 0$ and $y = g(v) \geq 0$. From the inequality (1) we get

$$f^t(v)\cdot(f(v)-g(v))+g^t(v)\cdot(g(v)-f(v))\geq 0,$$

true for all $v \in [a, b]$. Now using the Riemann integral we obtain the integral form of (1):

$$\int_a^b f^t(v)\cdot(f(v)-g(v))+g^t(v)\cdot(g(v)-f(v))dv\geq 0. \tag{2}$$

Conversely for fixed $x, y \geq 0$ positive real numbers let us choose the continuous functions $f, g : [a, b] \to [0, +\infty)$ such that $f(v) = x$ and $g(v) = y$ for all $v \in [a, b]$. Then from (2) we deduce

$$\int_a^b [x^t\cdot(x-y)+y^t\cdot(y-x)]dv\geq 0,$$

i.e.

$$[x^t \cdot (x-y) + y^t \cdot (y-x)] \cdot \int_a^b 1 dv \geq 0,$$

so we obtain (1).

The inequality (2) is equivalent with the following inequalities:

$$\int_a^b f^t(v)\cdot(f(v)-g(v))dv \geq \int_a^b g^t(v)\cdot(f(v)-g(v))dv,$$

or

$$\int_a^b f^{t+1}(v)dv + \int_a^b g^{t+1}(v)dv \geq$$

$$\geq \int_a^b f^t(v) \cdot g(v)dv + \int_a^b f(v) \cdot g^t(v)dv.$$

Taking the particular values for t we get the following.

For $t = 1$:

$$\int_a^b f^2(v)dv + \int_a^b g^2(v)dv \geq 2 \cdot \int_a^b f(v) \cdot g(v)dv,$$

which is also evident by direct calculus.

For $t = 2$ results:

$$\int_a^b f^3(v)dv + \int_a^b g^3(v)dv \geq$$

$$\geq \int_a^b f^2(v) \cdot g(v)dv + \int_a^b f(v) \cdot g^2(v)dv,$$

which is also true by direct calculus.

For $t = n \in \mathbb{N}, n \geq 3$ we obtain:

$$\int_a^b f^{n+1}(v)dv + \int_a^b g^{n+1}(v)dv \geq$$

$$\geq \int_a^b f^n(v) \cdot g(v)dv + \int_a^b f(v) \cdot g^n(v)dv.$$

Now we take particular functions f and g. For $0 \leq a < b$ and $v \in [a, b]$ we consider the functions $f(v) = e^v > 0$ and $g(v) = v \geq 0$ and we obtain the inequality:

$$\int_a^b e^{(n+1)\cdot v}dv + \int_a^b v^{n+1}dv \geq$$

$$\geq \int_a^b e^{nv} \cdot v\, dv + \int_a^b e^v \cdot v^n dv.$$

Now we calculate these integrals:

$$\int_a^b e^{(n+1)\cdot v}dv = \frac{e^{(n+1)\cdot b} - e^{(n+1)\cdot a}}{n+1},$$

$$\int_a^b v^{n+1}dv = \frac{b^{n+2} - a^{n+2}}{n+2},$$

$$\int_a^b e^{nv} \cdot v\, dv = \frac{e^{n\cdot b} \cdot b - e^{n\cdot a} \cdot a}{n} - \frac{e^{nb} - e^{na}}{n^2},$$

$$\int_a^b e^v \cdot v^n dv = e^b \cdot [b^n - n \cdot b^{n-1} + n(n-1) \cdot b^{n-2} +$$
$$+ \ldots + (-1)^{n-2} \cdot n(n-1) \cdot \ldots \cdot 3 \cdot b^2 +$$
$$+ (-1)^{n-1} \cdot n! \cdot b + (-1)^n \cdot n!] -$$
$$- e^a \cdot [a^n - n \cdot a^{n-1} + n(n-1) \cdot a^{n-2} +$$
$$\ldots + (-1)^{n-2} \cdot n(n-1) \cdot \ldots \cdot 3 \cdot a^2 +$$
$$(-1)^{n-1} \cdot n! \cdot a + (-1)^n \cdot n!],$$

and we obtain the inequality:

$$\frac{e^{(n+1)\cdot b} - e^{(n+1)\cdot a}}{n+1} + \frac{b^{n+2} - a^{n+2}}{n+2} \geq$$
$$\frac{e^{n\cdot b} \cdot b - e^{n\cdot a} \cdot a}{n} - \frac{e^{n\cdot b} - e^{n\cdot a}}{n^2} +$$
$$+ e^b \cdot [b^n - n \cdot b^{n-1} + n(n-1) \cdot b^{n-2} +$$
$$+ \ldots + (-1)^{n-2} \cdot n(n-1) \cdot \ldots \cdot 3 \cdot b^2 +$$
$$+ (-1)^{n-1} \cdot n! \cdot b + (-1)^n \cdot n!] -$$
$$- e^a \cdot [a^n - n \cdot a^{n-1} + n(n-1) \cdot a^{n-2} +$$
$$+ \ldots + (-1)^{n-2} \cdot n(n-1) \cdot \ldots \cdot 3 \cdot a^2 +$$
$$+ (-1)^{n-1} \cdot n! \cdot a + (-1)^n \cdot n!].$$

If we choose $a = 0$ and $b = 1$ we can deduce the inequality:

$$\frac{e^{n+1} - 1}{n+1} + \frac{1}{n+2} \geq \frac{e^n}{n} - \frac{e^n - 1}{n^2} +$$
$$+ e \cdot [1 - n + n(n-1) + \ldots +$$
$$+ (-1)^{n-2} \cdot n(n-1) \cdot \ldots \cdot 3 +$$
$$+ (-1)^{n-1} \cdot n! + (-1)^n \cdot n!] + (-1)^{n+1} \cdot n!$$

Now we rearrange the terms of this inequality in order to obtain an inequality for the exponential function:

$$\frac{e \cdot e^n}{n+1} - \frac{e^n}{n} + \frac{e^n}{n^2} \geq$$
$$\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n^2} +$$
$$+ e \cdot [1 - n + n(n-1) + \ldots +$$
$$+ (-1)^{n-2} \cdot n(n-1) \cdot \ldots \cdot 3 +$$
$$+ (-1)^{n-1} \cdot n! + (-1)^n \cdot n!] + (-1)^{n+1} \cdot n!$$

or

$$e^n \cdot \frac{en^2 - n^2 + 1}{(n+1)n^2} \geq \frac{2n^2 + 3n + 2}{n^2(n+1)(n+2)} +$$
$$+ e \cdot [1 - n + n(n-1) + \ldots +$$
$$+ (-1)^{n-2} \cdot n(n-1) \cdot \ldots \cdot 3 +$$
$$+ (-1)^{n-1} \cdot n! + (-1)^n \cdot n!] + (-1)^{n+1} \cdot n!$$

and finally we have

$$e^n \geq \frac{(n+1)n^2}{en^2 - n^2 + 1} \cdot \{\frac{2n^2 + 3n + 2}{n^2(n+1)(n+2)} +$$
$$+ e \cdot [1 - n + n(n-1) + \ldots +$$
$$+ (-1)^{n-2} \cdot n(n-1) \cdot \ldots \cdot 3 +$$
$$+ (-1)^{n-1} \cdot n! + (-1)^n \cdot n!] + (-1)^{n+1} \cdot n!\}$$

Let $\alpha : [0, +\infty) \to [0, +\infty)$ be a continuous and monotone increasing function. Then the inequality (1) becomes:

$$\alpha(x) \cdot (x - y) + \alpha(y) \cdot (y - x) \geq 0.$$

The proof is immediatly and for $\alpha(x) = x^t$ we reobtain (1).

We mention that we can choose more generally the function $\alpha : \mathbb{R} \to \mathbb{R}$ monotone increasing. In this case we have

$$\alpha(x) \cdot (x - y) + \alpha(y) \cdot (y - x) \geq 0,$$

for every $x, y \in \mathbb{R}$.

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous and monotone increasing function and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. If $a, b \in \mathbb{R}, a < b$, for every $v \in [a, b]$ we choose $x = f(v) \in \mathbb{R}$ and $y = g(v) \in \mathbb{R}$ and we can deduce:

$$\alpha(f(v)) \cdot (f(v) - g(v)) + \alpha(g(v)) \cdot (g(v) - f(v)) \geq 0$$

for every $v \in [a, b]$. Now we take the Riemann integral and we obtain the following integral form:

$$\int_a^b \alpha(f(v)) \cdot (f(v) - g(v)) + \alpha(g(v)) \cdot (g(v) - f(v)) dv \geq 0,$$

or

$$\int_a^b \alpha(f(v)) \cdot (f(v) - g(v)) dv \geq \int_a^b \alpha(g(v)) \cdot (f(v) - g(v)) dv,$$

or

$$\int_a^b \alpha(f(v)) \cdot f(v) dv + \int_a^b \alpha(g(v)) \cdot g(v) dv \geq$$
$$\geq \int_a^b \alpha(f(v)) \cdot g(v) dv + \int_a^b f(v) \cdot \alpha(g(v)) dv.$$

Now we can take particular functions for α, f and g. For $\alpha : \mathbb{R} \to \mathbb{R}, \alpha(x) = x^{2n+1}, n \in \mathbb{N}$ we get

$$\int_a^b f^{2n+2}(v) dv + \int_a^b g^{2n+2}(v) dv \geq$$
$$\geq \int_a^b f^{2n+1}(v) \cdot g(v) dv + \int_a^b f(v) \cdot g^{2n+1}(v) dv.$$

For $\alpha : \mathbb{R} \to \mathbb{R}, \alpha(x) = c^x$, where $c > 1$ we have

$$\int_a^b c^{f(v)} \cdot f(v) dv + \int_a^b c^{g(v)} \cdot g(v) dv \geq$$
$$\geq \int_a^b c^{f(v)} \cdot g(v) dv + \int_a^b f(v) \cdot c^{g(v)} dv.$$

If the function $\alpha : \mathbb{R} \to \mathbb{R}$ is monotone decreasing, then

$$\alpha(x) \cdot (x - y) + \alpha(y) \cdot (y - x) \leq 0,$$

for every $x, y \in \mathbb{R}$.

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a continuous and monotone decreasing function and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. If $a, b \in \mathbb{R}, a < b$, for every $v \in [a, b]$ we choose $x = f(v) \in \mathbb{R}$ and $y = g(v) \in \mathbb{R}$ and we can deduce:

$$\alpha(f(v)) \cdot (f(v) - g(v)) + \alpha(g(v)) \cdot (g(v) - f(v)) \leq 0$$

for every $v \in [a, b]$. Now we take the Riemann integral and we obtain the following integral form:

$$\int_a^b \alpha(f(v)) \cdot (f(v) - g(v)) + \alpha(g(v)) \cdot (g(v) - f(v)) dv \leq 0,$$

or

$$\int_a^b \alpha(f(v)) \cdot (f(v) - g(v))dv$$
$$\leq \int_a^b \alpha(g(v)) \cdot (f(v) - g(v))dv,$$

or

$$\int_a^b \alpha(f(v)) \cdot f(v)dv + \int_a^b \alpha(g(v)) \cdot g(v)dv \leq$$
$$\leq \int_a^b \alpha(f(v)) \cdot g(v)dv + \int_a^b f(v) \cdot \alpha(g(v))dv.$$

For the particular function $\alpha : \mathbb{R} \to \mathbb{R}, \alpha(x) = c^x$, where $c \in (0,1)$ we have

$$\int_a^b c^{f(v)} \cdot f(v)dv + \int_a^b c^{g(v)} \cdot g(v)dv \leq$$
$$\leq \int_a^b c^{f(v)} \cdot g(v)dv + \int_a^b f(v) \cdot c^{g(v)}dv.$$

3 Discussion and conclusion

In the next paper we will consider the original Schur inequality with four variables $x, y, z, t \geq 0$,

$$x^t \cdot (x-y) \cdot (x-z) + y^t \cdot (y-x)(y-z) + z^t (z-x)(z-y) \geq 0,$$

and we will give the integral form of this inequality and after we will deduce some applications.

References

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